Some (nontrivial) examples of vertex (operator) algebras

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Keywords:

A. Introduction

The goal of these notes is to give two interesting examples of vertex algebras. (By interesting, we mean to exclude examples that are vertex algebras as an indirect consequence of being some other nice algebraic structure. For example, any commutative associative algebra with identity is a vertex algebra. See remark 3.1.6 of [1].) We will prove a reconstruction theorem that allows us to generate vertex algebras from a suitable starting point, and we give two examples of vertex algebras that can be generated thus. Our treatment mostly follows [2].

First, we recall the definition of a vertex algebra:

Notation 1. For a (doubly-infinite) formal Laurent series $A(x) = \sum_k a_k x^{-k-1}$, we write

$$\partial_x A(x) = \partial A(x) = \sum_k -a_k(k+1)x^{-k-2}, \qquad \partial^m = \underbrace{\partial \circ \partial \circ \cdots \circ \partial}_{m \text{ times}}, \qquad D^m := \frac{\partial^m}{m!}. \tag{1}$$

Definition 2. A vertex algebra $(\mathsf{V}, Y, \mathbf{1})$ is a vector space V (here, with ground field $\mathbb{C}$) together with a distinguished vector $\mathbf{1} \in \mathsf{V}$ and a linear map

$$Y(\,\cdot\,, x) : \mathsf{V} \longrightarrow (\operatorname{End}\mathsf{V})[[x^{\pm 1}]], \qquad Y(v,x) = \sum_{n\in\mathbb{Z}} v_n x^{-n-1}, \qquad v_n \in \operatorname{End}\mathsf{V}, \qquad v \in \mathsf{V}, \tag{2}$$

(with x a formal variable), such that the following is true for all $u, v \in \mathsf{V}$:

1. *field property:* $Y(v,x)$ is a field (i.e., $Y(v,x)u \in \mathsf{V}((x))$.)
2. *vaccum property:* $Y(\mathbf{1}, x) = \mathrm{id}_{\mathsf{V}}$ (i.e., $\mathbf{1}_{-1} = \mathrm{id}_{\mathsf{V}}$ and $\mathbf{1}_m = 0$ if $m \neq -1$).
3. *state-field correspondence:* $\lim_{x\to 0} Y(v,x)\mathbf{1} = v$.
4. The *Jacobi condition* (see (3.1.6) of [1]), but by theorem 3.5.1 of [1], this is equivalent to the following two conditions:
   (a) *locality*, or *weak commutativity*: $Y(u,x)$ and $Y(v,x)$ are *mutually local*. That is, there exists an $N \in \mathbb{Z}^+$ (possibly depending on u and v) such that
   $$(x-y)^N [Y(u,x), Y(v,y)] = 0. \tag{3}$$
   (b) Let $\mathcal{D} \in \operatorname{End}\mathsf{V} \subset (\operatorname{End}\mathsf{V})[[x^{\pm 1}]]$ be defined by $\mathcal{D}(v) = v_{-2}\mathbf{1}$. Then
   $$[\mathcal{D}, Y(v,x)] = \partial Y(v,x). \tag{4}$$

B. The reconstruction theorem

In order to state the mentioned reconstruction theorem, we need to define the normal-ordered product of two (doubly-infinite) formal Laurent series and prove some useful lemmas regarding them. For this, we denote

$$A(x) \in (\operatorname{End}\mathsf{V})[[x^{\pm 1}]], \qquad A(x) = \sum_n A_n x^{-n-1}, \tag{5}$$

$$\Longrightarrow \qquad A(x)_- := \sum_{n\geq 0} A_n x^{-n-1}, \qquad A(x)_+ := \sum_{n<0} A_n x^{-n-1}. \tag{6}$$

Lemma 3. *Suppose that $A(x), B(x) \in (\operatorname{End}\mathsf{V})[[x^{\pm 1}]]$ are fields. Then $A(x)_+B(x)$ and $B(x)A(x)_-$ exist and are fields.*

Proof. Throughout, we write $A(x) = \sum_k A_k x^{-k-1}$ and $B(x) = \sum_l B_l x^{-l-1}$. Then because $A(x)$ and $B(y)$ are fields, for each $v \in \mathsf{V}$, there exist integers $n_A = n_A(v)$ and $l_B = l_B(v)$ such that

$$A_k(v) = 0 \text{ for all } k \geq k_A, \qquad \text{and} \qquad B_l(v) = 0 \text{ for all } l \geq l_B. \tag{7}$$

Thus, we have

$$A(x)_+ B(y)v = \sum_{k<0} \sum_{l<l_B} A_k B_l v\, x^{-k-1} y^{-l-1} \tag{8}$$

$$B(y)A(x)_- v = \sum_{0 \leq k < k_A} \sum_{l \in \mathbb{Z}} B_l A_k v\, x^{-k-1} y^{-l-1}. \tag{9}$$

Setting $y = x$ (and $m = k + l + 1$) then gives two distinct elements of $\mathsf{V}[[x^{\pm 1}]]$:

$$A(x)_+ B(x)v = \sum_{m \in \mathbb{Z}} \sum_{m - l_B - 1 < k < 0} (A_k B_{m-k-1}) v\, x^{-m-1} \tag{10}$$

$$B(x)A(x)_- v = \sum_{m \in \mathbb{Z}} \sum_{0 \leq k < k_A} B_{m-k-1} A_k v\, x^{-m-1}. \tag{11}$$

Thus, $A(x)_+ B(x) \in (\operatorname{End} \mathsf{V})[[x^{\pm 1}]]$ and $B(x)A(x)_- \in (\operatorname{End} \mathsf{V})[[x^{\pm 1}]]$. To see that $A(x)_+ B(x)$ is a field, we note that the coefficient of x^{-m-1} in (10) is zero if $m \geq l_B$. To see that $B(x)A(x)_-$ is a field, we note that because $B(x)$ is a field, $B_{m-k-1} A_k v = 0$ for all $m \in \mathbb{Z}^+$, $0 \leq k \leq k_A$ such that $m - k - 1 \geq l_B(A_k)$. This condition is satisfied if

$$m \geq \max\{l_B(A_k v) \,|\, 0 \leq k < k_A\} + k_A, \tag{12}$$

for then we have $m - k - 1 \geq m - (k_A - 1) - 1 \geq \max\{l_B(A_k v) \,|\, 0 \leq k < k_A\} \geq l_B(A_k v)$. □

Lemma 4. *Suppose that $A(x) \in (\operatorname{End} \mathsf{V})[[x^{\pm 1}]]$. Then we have $(\partial A(x))_\pm = \partial A(x)_\pm$.*

Proof. Easy. □

Definition 5. The *normal-ordered product* of two formal Laurent series $A(x), B(x) \in (\operatorname{End} \mathsf{V})[[x^{\pm 1}]]$ is

$$:A(x)B(y): := A(x)_+ B(y) + B(y)A(x)_- \in (\operatorname{End} \mathsf{V})[[x^{\pm 1}, y^{\pm 1}]]. \tag{13}$$

The normal-ordered product of n formal Laurent series $A_1(x)$, $A_2(x), \ldots, A_n(x) \in (\operatorname{End} \mathsf{V})[[x^{\pm 1}]]$ is

$$:A_1(x_1)A_2(x_2) \ldots A_n(x_n): := \\ :A_1(x_1) :A_2(x_2) :A_3(x_3) \cdots :A_{n-2}(x_{n-2}) :A_{n-1}(x_{n-1})A_n(x_n):: \cdots ::: \in (\operatorname{End} \mathsf{V})[[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]]. \tag{14}$$

Lemma 6. *Suppose that $A(x), B(x) \in (\operatorname{End} \mathsf{V})[[x^{\pm 1}]]$ are fields. Then $:A(x)B(x):$ exists and is a field, and $\partial^m A(x)$ is a field for any $m \in \mathbb{Z}^+$.*

Proof. The first statement is an immediate consequence of (13) and lemma 9. The second is obvious. □

In particular, it follows from lemma 6 that if $A_1(x)$, $A_2, (x), \ldots, A_n(x) \in (\operatorname{End} \mathsf{V})[[x^{\pm 1}]]$ are fields, then the normal-ordered product $:A_1(x)A_2(x) \ldots A_n(x):$ is a field too. Also, (10, 11), summed together, give us this useful formula:

$$:A(x)B(x):= \sum_{m \in \mathbb{Z}} \left(\sum_{k<0} A_k B_{m-k-1} + \sum_{k \geq 0} B_{m-k-1} A_k \right) x^{-m-1}. \tag{15}$$

Lemma 7. *Suppose that $D : (\operatorname{End} \mathsf{V})[[x^{\pm 1}]] \longrightarrow (\operatorname{End} \mathsf{V})[[x^{\pm 1}]]$ is a derivation satisfying $(DA(x))_\pm = D(A(x)_\pm)$. Then D is also a derivation with respect to the normal-ordered product of two fields. That is,*

$$D(:A(x)B(x):) =:DA(x)B(x): + :A(x)DB(x): . \tag{16}$$

Proof. We have

$$D(:A(x)B(x):) = D(A(x)_+ B(x)) + D(B(x)A(x)_-) \tag{17}$$

$$= D(A(x)_+)B(x) + A(x)_+(DB(x)) + (DB(x))A(x)_- + B(x)D(A(x)_-) \tag{18}$$

$$=:DA(x)B(x): + :A(x)DB(x):, \tag{19}$$

which conclude the proof. □

Lemma 8. (Dong's lemma) *Suppose that* $A(x), B(x), C(x) \in (\mathrm{End}\,\mathsf{V})[[x^{\pm 1}]]$ *are pairwise mutually local. Then* $:\!A(x)B(x)\!:$ *and* $C(x)$ *are mutually local.*

Proof. This is the proof given in [3]. Because $A(x) = \sum_k A_k x^{-k-1}$, $B(x) = \sum_l B_l x^{-l-1}$, and $C(x) = \sum_m C_m x^{-m-1}$ are all pairwise mutually local, there exists an $N \in \mathbb{Z}^+$ such that

$$(x-y)^N [A(x), B(y)] = 0, \qquad (y-z)^N [B(y), C(z)] = 0, \qquad (x-z)^N [A(x), C(z)] = 0. \tag{20}$$

Now with $\iota_{x,y}(x-y)^{-1}$ and $\iota_{y,x}(x-y)^{-1}$ denoting the formal Laurent series given by expanding the rational function $(x-y)^{-1}$ in the domain $|x| > |y|$ and $|y| > |x|$ respectively, we define

$$F(x,y,z) := \iota_{x,y}(x-y)^{-1} A(x)B(y)C(z) - \iota_{y,x}(x-y)^{-1} B(y)A(x)C(z) \tag{21}$$
$$G(x,y,z) := \iota_{x,y}(x-y)^{-1} C(z)A(x)B(y) - \iota_{y,x}(x-y)^{-1} C(z)B(y)A(x). \tag{22}$$

The residues of these formal Laurent series (21, 22) at $x = 0$ show why we consider these series in the first place. For example, we have (with $n = k + l + 1$ and using (15))

$$F(x,y,z) = \sum_{\substack{j \geq 0 \\ k,l,m \in \mathbb{Z}}} \left(A_k B_l C_m x^{-j-k-2} y^{j-l-1} + B_l A_k C_m x^{j-k-1} y^{-j-l-2} \right) z^{-m-1} \tag{23}$$

$$\Longrightarrow \qquad \mathrm{Res}_{x=0} F(x,y,z) = \sum_{m,n \in \mathbb{Z}} \left(\sum_{k<0} A_k B_{n-k-1} + \sum_{k \geq 0} B_{n-k-1} A_k \right) C_m y^{-n-1} z^{-m-1} \tag{24}$$
$$= :\!A(y)B(y)\!:\, C(z). \tag{25}$$

Similarly, after pushing the C_m to the left of the $A_k B_{n-k-1}$, we find that $\mathrm{Res}_{x=0} G(x,y,z) = C(z) :\!A(y)B(y)\!:$ too. Thus, to prove that $:\!A(y)B(y)\!:$ and $C(z)$ are mutually local, it suffices to prove that

$$(y-z)^{3N} \big(F(x,y,z) - G(x,y,z) \big) = 0 \tag{26}$$

and take the residue centered on $x = 0$ of this result (26). Now to prove (26), we consider the binomial expansion

$$(y-z)^{3N} = \overbrace{\sum_{p=0}^{N-1} \binom{2N}{p} (y-x)^{2N-p} (x-z)^p (y-z)^N}^{:=\alpha(x,y,z)} + \overbrace{\sum_{p=N}^{2N} \binom{2N}{p} (y-x)^{2N-p} (x-z)^p (y-z)^N}^{:=\beta(x,y,z)}. \tag{27}$$

Because the power $2N - p$ of $(y - x)$ in $\alpha(x,y,z)$ is greater than N, multiplying (21, 22) by $\alpha(x,y,z)$ and invoking (20) gives (of course, if $p \in \mathbb{Z}^+$, then $(y-x)^p \iota_{x,y}(x-y)^{-1} = (y-x)^p \iota_{y,x}(x-y)^{-1} = (-1)^p (x-y)^{p-1}$)

$$\alpha(x,y,z) F(x,y,z) = \sum_{p=0}^{N-1} \binom{2N}{p} (-1)^{2N-p-1} (x-z)^p (y-z)^N (x-y)^{2N-p-1} [A(x), B(y)] C(z) = 0, \tag{28}$$

$$\alpha(x,y,z) G(x,y,z) = \sum_{p=0}^{N-1} \binom{2N}{p} (-1)^{2N-p-1} (x-z)^p (y-z)^N C(z) (x-y)^{2N-p-1} [A(x), B(y)] = 0. \tag{29}$$

Thus, we have

$$\begin{aligned} (y-z)^{3N} \big(F(x,y,z) - G(x,y,z) \big) &= \beta(x,y,z) \big(F(x,y,z) - G(x,y,z) \big) \\ &= \sum_{p=N}^{2N} \binom{2N}{p} (-1)^{2N-p-1} \big(\iota_{x,y}(x-y)^{2N-p-1} (y-z)^N (x-z)^p [A(x)B(y), C(z)] \\ &\qquad - \iota_{y,x}(x-y)^{2N-p-1} (y-z)^N (x-z)^p [C(z), A(x)B(y)] \big). \end{aligned} \tag{30}$$

By writing the commutators in (30) in terms of the two commutators $[A(x), C(z)]$ and $[B(y), C(z)]$ and allowing $(x-z)^p$ and $(y-z)^N$ to respectively act on these by left multiplication, we conclude from (20) that (30) vanishes, which is what (26) we sought to prove.

□

We note one more easy fact before stating the reconstruction theorem.

Lemma 9. *Suppose that $A(x), B(x) \in (\mathrm{End}\,\mathsf{V})[[x^{\pm 1}]]$ are mutually local, i.e., there exists an $N \in \mathbb{Z}^+$ such that*

$$(x-y)^N [A(x), B(y)] = 0. \tag{31}$$

Then for any $m, n \in \mathbb{Z}^+ \cup \{0\}$, $\partial^m A(x)$ and $\partial^n B(x)$ are mutually local.

Proof. It suffices to prove the theorem for $m = 1$ and $n = 0$ and for $m = 0$ and $n = 1$. We prove only the former, and the proof of the latter is obtained by replacing $\partial_x \mapsto \partial_y$. Differentiating (31) with respect to x gives

$$0 = \partial_x\big((x-y)^N [A(x), B(y)]\big) = N(x-y)^{N-1}[A(x), B(y)] + (x-y)^N [\partial A(x), B(y)]. \tag{32}$$

Multiplying this by $(x-y)$ and applying (31), we find that $(x-y)^{N+1}[\partial A(x), B(y)] = 0$, so $\partial A(x)$ and $B(y)$ are mutually local. □

Now we are ready to state and prove the reconstruction theorem (theorem 10.24 of [2]). For this, we introduce the following notation: for a collection $\{\varphi_a(x) \,|\, a \in \mathsf{V}\} \subset (\mathrm{End}\,\mathsf{V})[[x^{\pm 1}]]$, we let

$$J^{a^{(1)}, a^{(2)}, \ldots, a^{(n)}}_{m_1, m_2, \ldots, m_n}(x) := :D^{m_1 - 1}\varphi_{a^{(1)}}(x) D^{m_2 - 1}\varphi_{a^{(2)}}(x) \cdots D^{m_n - 1}\varphi_{a^{(n)}}(x):, \qquad m_k, n \in \mathbb{Z}^+, \quad a^{(i)} \in \mathsf{V}. \tag{33}$$

Theorem 10. (The weak reconstruction theorem) *Suppose that V is a vector space (over $\mathbb{C}$) with a distinguished vector $\mathbf{1} \in \mathsf{V}$ and $\mathsf{I} \subset V$ is linearly independent. Suppose further that $\{\varphi_a(x) \,|\, a \in \mathsf{I}\} \subset (\mathrm{End}\,\mathsf{V})[[x^{\pm 1}]]$, writing*

$$\varphi_a(x) := \sum_k a_k x^{-k-1} \in (\mathrm{End}\,\mathsf{V})[[x^{\pm 1}]], \qquad a \in \mathsf{I}, \tag{34}$$

and let $T \in \mathrm{End}\,\mathsf{V} \subset (\mathrm{End}\,\mathsf{V})[[x^{\pm 1}]]$. If the following are true for all $a, b \in \mathsf{I}$,

- *1′. $\varphi_a(x)$ is a field,*
- *3′. $\lim_{x \to 0} \varphi_a(x)\mathbf{1} = a$,*
- *4′. the following two conditions,*
  - *(a) $\varphi_a(x)$ and $\varphi_b(x)$ are mutually local,*
  - *(b) $[T, \varphi_a(x)] = \partial\varphi_a(x)$ and $T\mathbf{1} = 0$,*
- *5′. the following set is a basis for V (note that we may have $a^{(i)} = a^{(j)}$ for some $i, j \in \mathbb{Z}^+$),*

$$\beta = \big\{a^{(1)}_{-m_1} a^{(2)}_{-m_2} \cdots a^{(n)}_{-m_n}\mathbf{1} \,\big|\, a^{(i)} \in \mathsf{I} \text{ and } n, m_k \in \mathbb{Z}^+\big\} \cup \{\mathbf{1}\}, \tag{35}$$

then $(\mathsf{V}, Y, \mathbf{1})$ is a vertex algebra, where $Y(\,\cdot\,, x) : \mathsf{V} \longrightarrow (\mathrm{End}\,\mathsf{V})[[x^{\pm 1}]]$ is defined by (33)

$$Y(a^{(1)}_{-m_1} a^{(2)}_{-m_2} \cdots a^{(n)}_{-m_n}\mathbf{1}, x) = J^{a^{(1)}, a^{(2)}, \ldots, a^{(n)}}_{m_1, m_2, \ldots, m_n}(x), \qquad Y(\mathbf{1}, x) = \mathrm{id}_{\mathsf{V}}. \tag{36}$$

Furthermore, $T = \mathcal{D}$, and $Y(a, x) = \varphi_a(x)$ for all $a \in \mathsf{I}$.

Proof. Because (35) is a basis, Y is well-defined by (36). Hence, what remains is to prove that the collection $(\mathsf{V}, Y, \mathbf{1})$ satisfies properties 1–4 of definition 2.

1. It immediately follows from lemmas 6 (see also the comment beneath lemma 6), (36), and linearity of Y that $Y(v, x)$ is a field for all $v \in \mathsf{V}$. Thus, $(\mathsf{V}, Y, \mathbf{1})$ satisfies property 1 of definition 2.
2. It is stated in (36) that $Y(\mathbf{1}, x) = \mathrm{id}_{\mathsf{V}}$. Hence, $(\mathsf{V}, Y, \mathbf{1})$ satisfies property 2 of definition 2.

3. We prove that $(\mathsf{V}, Y, \mathbf{1})$ satisfies property 3 of definition 2, $\lim_{x\to 0} Y(v,x)\mathbf{1} = v$, for any $v \in \beta$, and thus by linear extension, for any $v \in \mathsf{V}$. First, if $v = \mathbf{1}$, then (36) gives

$$\lim_{x\to 0} Y(\mathbf{1},x)\mathbf{1} = \lim_{x\to 0} \mathbf{1} = \mathbf{1}. \tag{37}$$

Next, if $v \neq \mathbf{1}$, then $v = a^{(1)}_{-m_1} a^{(2)}_{-m_2} \cdots a^{(n)}_{-m_n}\mathbf{1}$ for some $n, m_k \in \mathbb{Z}^+$ and $a^{(i)} \in \mathsf{I}$, so $Y(v,x) = J^{a^{(1)},a^{(2)},\ldots,a^{(n)}}_{m_1,m_2,\ldots,m_n}(x)$ by (36). Thus, proving that $(\mathsf{V}, Y, \mathbf{1})$ satisfies property 3 of definition 2 for such v amounts to showing that

$$\lim_{x\to 0} J^{a^{(1)},a^{(2)},\ldots,a^{(n)}}_{m_1,m_2,\ldots,m_n}(x)\mathbf{1} = a^{(1)}_{-m_1} a^{(2)}_{-m_2} \cdots a^{(n)}_{-m_n}\mathbf{1}. \tag{38}$$

We prove (38) by induction on n. First assuming that $n = 1$, we have for any $m \in \mathbb{Z}^+$ that

$$J^a_m(x) = D^{m-1}\varphi_a(x) = (-1)^{m-1}\sum_k \binom{k+m-1}{m-1} a_k x^{-k-m}. \tag{39}$$

The binomial coefficient in (39) vanishes for $k \in \{-1,-2,\ldots,-(m-1)\}$, and item $3'$ of the lemma statement implies that $a_k\mathbf{1} = 0$ for $k \geq 0$ and $a_{-1}\mathbf{1} = a$. Thus, we have

$$J^a_m(x)\mathbf{1} = (-1)^{m-1}\sum_{k<-(m-1)} \binom{k+m-1}{m-1} a_k\mathbf{1}\, x^{-k-m}, \tag{40}$$

$$\implies \qquad \lim_{x\to 0} J^a_m(x)\mathbf{1} = (-1)^{m-1}\binom{-1}{m-1} a_{-m}\mathbf{1} = a_{-m}\mathbf{1}. \tag{41}$$

This proves (38) for $n = 1$. Now, assuming that (38) holds for $n \mapsto n-1$, we prove that (38) holds for n as well. Using (33), we may write

$$J^{a^{(1)},a^{(2)},\ldots,a^{(n)}}_{m_1,m_2,\ldots,m_n}(x) =:A(x)B(x):, \qquad A(x) := J^{a^{(1)}}_{m_1}(x), \quad B(x) := J^{a^{(2)},a^{(3)},\ldots,a^{(n)}}_{m_2,m_3,\ldots,m_n}(x). \tag{42}$$

Now from (41, 42) and the induction assumption, we respectively have

$$\lim_{x\to 0} A(x)\mathbf{1} = a^{(1)}_{-m_1}\mathbf{1}, \qquad \implies \qquad A_n\mathbf{1} = 0, \quad n \geq 0, \qquad A_{-1}\mathbf{1} = a^{(1)}_{-m_1}\mathbf{1}, \tag{43}$$

$$\lim_{x\to 0} B(x)\mathbf{1} = a^{(2)}_{-m_2} a^{(3)}_{-m_3} \cdots a^{(m)}_{-m_n}\mathbf{1} \qquad \implies \qquad B_n\mathbf{1} = 0, \quad n \geq 0, \qquad B_{-1}\mathbf{1} = a^{(2)}_{-m_2} a^{(3)}_{-m_3} \cdots a^{(m)}_{-m_n}\mathbf{1}. \tag{44}$$

The conditions that $A_n\mathbf{1} = 0$ and $B_n\mathbf{1} = 0$ for $n \geq 0$ imply that $A(x)_-\mathbf{1} = 0$ and $B(x)_-\mathbf{1} = 0$ respectively. Hence, we have (6)

$$\lim_{x\to 0} :A(x)B(x):\,\mathbf{1} = \lim_{x\to 0}\big(A(x)_+B(x)_+\mathbf{1} + A(x)_+B(x)_-\mathbf{1} + B(x)A(x)_-\mathbf{1}\big) \tag{45}$$

After inserting $A(x)_-\mathbf{1} = 0 = B(x)_-\mathbf{1} = 0$ into (45), sending $x \to 0$, and inserting (44) into the result, we obtain

$$\begin{aligned}\lim_{x\to 0} :A(x)B(x):\,\mathbf{1} &= \lim_{x\to 0} A(x)_+B(x)_+\mathbf{1} \\ &= A_{-1}B_{-1}\mathbf{1} = A_{-1}a^{(2)}_{-m_2} a^{(3)}_{-m_3} \cdots a^{(m)}_{-m_n}\mathbf{1}.\end{aligned} \tag{46}$$

Now to find A_{-1}, we use (39) with (42) to write

$$A(x) = (-1)^{m_1-1}\sum_l \binom{l}{m_1-1} a^{(1)}_{l-m_1+1} x^{-l-1} \implies \qquad A_{-1} = (-1)^{m_1-1}\binom{-1}{m_1-1} a^{(1)}_{-m_1} = a^{(1)}_{-m_1}. \tag{47}$$

After inserting (42, 47) into the left and right sides of (46) respectively, we arrive with (38). Hence, $(\mathsf{V}, Y, \mathbf{1})$ satisfies property 3 of definition 2.

4. Next, we prove that $(\mathsf{V}, Y, \mathbf{1})$ satisfies property 4 of definition 2.

   (a) It immediately follows from lemmas 8 and 9 that the fields $J^{a^{(1)},a^{(2)},\ldots,a^{(n)}}_{m_1,m_2,\ldots,m_n}(x)$ (33) are pairwise mutually local. As such, it immediately follows from (36) that $Y(u,x)$ and $Y(v,x)$ are mutually local for all $u, v \in \beta$. By linearity, this extends to all $u, v \in \mathsf{V}$. Thus, $(\mathsf{V}, Y, \mathbf{1})$ satisfies property 4a of definition 2.

(b) To prove that $(\mathsf{V}, Y, \mathbf{1})$ satisfies property 4b of definition 2, we first prove that $[T, Y(v,x)] = \partial Y(v,x)$, for any $v \in \beta$, and by linear extension, for any $v \in \mathsf{V}$, and then we prove that $T = \mathcal{D}$. First, if $v = \mathbf{1}$, then (36) gives

$$[T, Y(\mathbf{1}, x)] = [T, \mathrm{id}_{\mathsf{V}}] = 0 = \partial(\mathrm{id}_{\mathsf{V}}) = \partial Y(\mathbf{1}, x). \tag{48}$$

Next, if $v \neq \mathbf{1}$, then $v = a^{(1)}_{-m_1} a^{(2)}_{-m_2} \cdots a^{(n)}_{-m_n} \mathbf{1}$ for some $n, m_k \in \mathbb{Z}^+$ and $a^{(i)} \in \mathsf{I}$, so $Y(v,x) = J^{a^{(1)},a^{(2)},\ldots,a^{(n)}}_{m_1,m_2,\ldots,m_n}(x)$ by (36). Thus, proving that $(\mathsf{V}, Y, \mathbf{1})$ satisfies property 4b of definition 2 for such v amounts to showing that

$$[T, J^{a^{(1)},a^{(2)},\ldots,a^{(n)}}_{m_1,m_2,\ldots,m_n}(x)] = \partial J^{a^{(1)},a^{(2)},\ldots,a^{(n)}}_{m_1,m_2,\ldots,m_n}(x). \tag{49}$$

We prove (49) by induction on n. First assuming that $n = 1$, noting that $TD^{m-1}\varphi_a(x) = D^{m-1}T\varphi_a(x)$, and recalling (39) and the first property of item $4'$b in the lemma statement, we find that for any $m \in \mathbb{Z}^+$,

$$\begin{aligned}
[T, J^a_m(x)] = [T, D^{m-1}\varphi_a(x)] &= TD^{m-1}\varphi_a(x) - D^{m-1}\varphi_a(x)T &(50)\\
&= TD^{m-1}\varphi_a(x) - D^{m-1}T\varphi_a(x) + D^{m-1}[T, \varphi_a(x)] &(51)\\
&= 0 + D^{m-1}\partial\varphi_a(x) = \partial J^a_m(x). &(52)
\end{aligned}$$

This proves (49) for $n = 1$. Now, assuming that (49) holds for $n \mapsto n-1$, we prove that (49) holds for n as well. Here, we use the notation of (42) again. Because $\operatorname{ad} T$ and ∂ are derivations, lemma 7 implies that

$$\begin{aligned}
[T, {:}A(x)B(x){:}] &= {:}[T, A(x)]B(x){:} + {:}A(x)[T, B(x)]{:}\\
&= {:}\partial A(x)B(x){:} + {:}A(x)\partial B(x){:}\\
&= \partial\, {:}A(x)B(x){:}\,.
\end{aligned} \tag{53}$$

After inserting (42) into the left and right sides of (53) respectively, we arrive with (49). Thus, $[T, Y(v,x)] = \partial Y(v,x)$ for all $v \in \mathsf{V}$.

To finish, we prove that $T = \mathcal{D}$. Here, the relation $[T, Y(v,x)] = \partial Y(v,x)$ with the second property $T\mathbf{1} = 0$ of item $4'$b gives the following for all $v \in \mathsf{V}$:

$$\begin{aligned}
& [T, Y(v,x)]\mathbf{1} = TY(v,x)\mathbf{1}\\
\Longrightarrow\quad & \partial Y(v,x)\mathbf{1} = TY(v,x)\mathbf{1}\\
\Longrightarrow\quad & \sum\nolimits_k -(k+1)v_k\mathbf{1}\, x^{-k-2} = Tv + O(x) \qquad \text{by item 3 of this proof,}\\
\Longrightarrow\quad & \mathcal{D}v := v_{-2}\mathbf{1} = Tv \qquad \Longrightarrow \qquad T = \mathcal{D}.
\end{aligned} \tag{54}$$

This proves that $(\mathsf{V}, Y, \mathbf{1})$ satisfies property 4b of definition 2 and that $T = \mathcal{D}$.

All that remains is to show that $Y(a,x) = \varphi_a(x)$ for all $a \in \mathsf{I}$. But from property $3'$, we have $a_{-1}\mathbf{1} = a$, so from this and (36), we conclude that $Y(a,x) = \varphi_a(x)$. □

With the help of the following lemma, we may relax item $5'$ of theorem 10 to say that β (35) simply spans V.

Lemma 11. (Goddard's uniqueness theorem) *Let $(\mathsf{V}, \mathbf{1}, Y)$ be a vertex algebra, and suppose that $A(x) \in (\operatorname{End}\mathsf{V})[[x^{\pm 1}]]$ is a field. If $A(x)$ and $Y(v,x)$ are mutually local for all $v \in \mathsf{V}$ and*

$$A(x)\mathbf{1} = Y(a,x)\mathbf{1} \tag{55}$$

for some $a \in \mathsf{V}$, then $A(x) = Y(a,x)$.

Proof. This proof is given in [4]. Let v be any vector in V. Because $A(x)$ and $Y(a,x)$ are both locally mutual with $Y(v,x)$, there exists $N \in \mathbb{Z}^+$ such that

$$(x-y)^N A(x)Y(v,y)\mathbf{1} = (x-y)^N Y(v,y)A(x)\mathbf{1} = (x-y)^N Y(v,y)Y(a,x)\mathbf{1} = (x-y)^N Y(a,x)Y(v,y)\mathbf{1}. \tag{56}$$

Because $\lim_{y\to 0} Y(v,y)\mathbf{1} = v$ by property 3 of definition 2, we may set $y = 0$ in (56) to find that

$$x^N A(x)v = x^N Y(a,x)v \tag{57}$$

for all $v \in \mathsf{V}$. Hence, we conclude that $A(x) = Y(a,x)$. □

Theorem 12. (The strong reconstruction theorem) *Suppose that all of the conditions of theorem 10 hold, except that condition 5′ is replaced by the weaker condition*

5″. the following set spans V *(note that we may have* $a^{(i)} = a^{(j)}$ *for some* $i, j \in \mathbb{Z}^+$*),*

$$\gamma = \left\{a^{(1)}_{-m_1} a^{(2)}_{-m_2} \cdots a^{(n)}_{-m_n} \mathbf{1} \,\middle|\, a^{(i)} \in \mathsf{I} \text{ and } n, m_k \in \mathbb{Z}^+\right\} \cup \{\mathbf{1}\}. \tag{58}$$

Then the conclusions of theorem 10 remain true after replacing condition 5′ of that theorem with condition 5″ here.

Proof. If γ (58) is a basis for V, then the statement of this theorem is equivalent to that of theorem 10. Hence, we assume otherwise. Then we choose a basis $\beta_1 \subset \gamma$ for V that contains $\mathbf{1}$, and we define

$$Y_1(a^{(1)}_{-m_1} a^{(2)}_{-m_2} \cdots a^{(n)}_{-m_n} \mathbf{1}, x) := J^{a^{(1)}, a^{(2)}, \ldots, a^{(n)}}_{m_1, m_2, \ldots, m_n}(x), \qquad a^{(1)}_{-m_1} a^{(2)}_{-m_2} \cdots a^{(n)}_{-m_n} \mathbf{1} \in \beta_1, \qquad Y_1(\mathbf{1}, x) = \mathrm{id}_{\mathsf{V}}. \tag{59}$$

Then for all $v \in \mathsf{V}$, $Y_1(v, x)$ follows from linear extension of (59). Now, item 1 in the proof of theorem 10 shows that $Y_1(v, x)$ is a field for all $v \in \mathsf{V}$, and that $Y(\mathbf{1}, x) = \mathrm{id}_{\mathsf{V}}$ is stated in (59). Also, (37, 38, 59) show that $\lim_{x\to 0} Y_1(v, x)\mathbf{1} = v$ for all $v \in \beta$ and, by linear extension, all $v \in \mathsf{V}$. Finally, (48, 49, 59) show that $[T, Y_1(v, x)] = \partial Y_1(v, x)$ for all $v \in \beta$ and, again by linear extension, all $v \in \mathsf{V}$, and the work in (54) then shows that $T = \mathcal{D}$. Hence, we conclude that $(V, \mathbf{1}, Y_1)$ is a vertex algebra. Furthermore, if $\beta_2 \subset \gamma$ is another basis of V containing $\mathbf{1}$ and we define

$$Y_2(a^{(1)}_{-m_1} a^{(2)}_{-m_2} \cdots a^{(n)}_{-m_n} \mathbf{1}, x) := J^{a^{(1)}, a^{(2)}, \ldots, a^{(n)}}_{m_1, m_2, \ldots, m_n}(x), \qquad a^{(1)}_{-m_1} a^{(2)}_{-m_2} \cdots a^{(n)}_{-m_n} \mathbf{1} \in \beta_2, \qquad Y_2(\mathbf{1}, x) = \mathrm{id}_{\mathsf{V}}, \tag{60}$$

then the same reasoning shows that $(V, \mathbf{1}, Y_2)$ is a vertex algebra. Thus, to show that the conclusions of theorem 10 remain true after replacing its condition 5′ with condition 5″ of this theorem, we only need to show that $Y_1 = Y_2$. To prove this, we invoke Goddard's uniqueness theorem 11 with $Y = Y_2$ and $A(x) = Y_1(v, x)$ for some $v \in \mathsf{V}$. To use this theorem, we observe that for any $v, w \in \mathsf{V}$, $Y_1(v, x)$ and $Y_2(w, x)$, given by linear extension of (59, 60), are mutually local thanks to lemmas 8 and 9. Also, we have

$$Y_1(v, x)\mathbf{1} = Y_2(v, x)\mathbf{1} \qquad \text{for all } v \in \mathsf{V}. \tag{61}$$

Indeed, to see why (61) is true, we rewrite item 3 of definition 2 to say

$$Y_i(v, x)\mathbf{1} = \sum_{k<0} v_{i,k}\mathbf{1}\, x^{-k-1}, \qquad v_{i,k}\mathbf{1} = \lim_{x\to 0} D^{-k-1} Y_i(v, x)\mathbf{1}, \qquad v_{i,-1}\mathbf{1} = v, \qquad i \in \{1, 2\}, \quad k <\in \mathbb{Z}^-. \tag{62}$$

Therefore, what remains in order to prove that $Y_1(v, x)\mathbf{1} = Y_2(v, x)\mathbf{1}$ for all $v \in \mathsf{V}$ is to show that $v_{1,k}\mathbf{1} = v_{2,k}\mathbf{1}$ for all $k < -1$. Now, proposition 3.1.18 of [1] states that $\partial Y_i(v, x) = Y_i(\mathcal{D}v, x)$, and it is easy to extend this fact to $\partial^{-k-1} Y_i(v, x) = Y_i(\mathcal{D}^{-k-1} v, x)$ for all $k < -2$. Hence, we have

$$\begin{aligned} v_{1,k}\mathbf{1} = \lim_{x\to 0} D^{-k-1} Y_1(v, x)\mathbf{1} &= \frac{1}{(-k-1)!} \lim_{x\to 0} Y_1(\mathcal{D}^{-k-1} v, x)\mathbf{1} = \frac{1}{(-k-1)!} \mathcal{D}^{-k-1} v \\ &= \frac{1}{(-k-1)!} \lim_{x\to 0} Y_2(\mathcal{D}^{-k-1} v, x)\mathbf{1} = \lim_{x\to 0} D^{-k-1} Y_2(v, x)\mathbf{1} = v_{2,k}\mathbf{1}, \qquad k \in \mathbb{Z}^-. \end{aligned} \tag{63}$$

From this with (62), we conclude that (61) holds. Therefore, Goddard's uniqueness theorem 11 implies that $Y_1(v, x) = Y_2(v, x)$ for all $v \in \mathsf{V}$, or that $Y_1 = Y_2$. □

C. The Heisenberg vertex algebra

In this section, we consider the Heisenberg algebra H. This is the infinite dimensional complex Lie algebra with basis $\{a_k \,|\, k \in \mathbb{Z}\} \cup \{Z\}$ and commutation relations

$$[a_m, a_n] = m\delta_{n+m,0} Z, \qquad [Z, a_m] = 0, \qquad m, n \in \mathbb{Z}. \tag{64}$$

The complex vector space of polynomials in an infinite number of variables $\mathsf{S} = \mathbb{C}[t_1, t_2, \ldots]$ becomes an H-module under the (irreducible) representation $\rho : \mathsf{H} \longrightarrow \operatorname{End} \mathsf{S}$ given by the linear extension of

$$\rho(a_n) = \partial_{t_n} \text{ for } n > 0, \qquad \rho(a_0) = \mu\, \mathrm{id}_{\mathsf{S}} \text{ for } \mu \in \mathbb{C}, \qquad \rho(a_{-n}) = n t_n \text{ for } n > 0, \qquad \rho(Z) = \mathrm{id}_{\mathsf{S}}. \tag{65}$$

In this representation, a_{-n} and a_n for $n > 0$ are respectively called *creation* and *annihilation* operators. Any product of the former "creates" a non-constant monomial from the unit monomial $1 \in \mathsf{S}$:

$$a_{-m_1}a_{-m_2}\cdots a_{m_n}.1 = t_{m_1}t_{m_2}\cdots t_{m_n}. \tag{66}$$

Although the powers of the right side appear to all equal one, they in fact may be any positive integer. For example, if $m_1 = m_2 = n$, then on the right side of (66), the power of t_n is at least two. Thus,

$$\beta = \left\{a_{-m_1}a_{-m_2}\cdots a_{-m_n}.1 \,|\, n, m_k \in \mathbb{Z}^+\right\} \cup \{1\} \tag{67}$$

is a basis for S. In order to endow $\mathsf{V} = \mathsf{S}$ with a vertex algebra structure, we must identify a vacuum vector $\mathbf{1}$ and vertex operator $Y(v, x)$ for each $v \in \mathsf{S}$. We anticipate using the (weak) reconstruction theorem 10 to do this. In light of condition $5'$ with (67), it is natural to choose

$$\mathbf{1} = 1 \in \mathsf{S} \quad \text{(the unit polynomial)} \tag{68}$$

for the vacuum vector and to choose $\mathsf{I} \subset \mathsf{S}$ such that, for some $a \in \mathsf{I}$, the coefficients of $\varphi_a(x)$ (34) are the creation and annihilation operators of H (or really, their images under ρ). Then condition $3'$ of theorem 10 says that such $a \in \mathsf{I}$ satisfies

$$a = \lim_{x\to 0}\varphi_a(x)\mathbf{1} = \lim_{x\to 0}\sum_k (a_k.1)x^{-k-1} = a_{-1}.1 = t_1 \in \mathsf{S}. \tag{69}$$

We choose $\mathsf{I} = \{a = a_{-1}.t\}$, so $\varphi_a(x) = \sum_k \rho(a_k)x^{-k-1}$ as desired. (Abusing notation, we write a_k for $\rho(a_k)$.) Then because any polynomial in S is annihilated by a sufficiently large number of partial derivatives, $\varphi_a(x)$ is a field, so condition $1'$ of theorem 10 is satisfied. Also, (69) shows that condition $3'$ of this theorem is satisfied too. Next,

$$\begin{aligned}[\varphi_a(x), \varphi_a(y)] &= \sum_{k,l}[a_k, a_l]x^{-k-1}y^{-l-1} \\ &= \sum_{k,l} k\delta_{k+l,0}x^{-k-1}y^{-l-1} \\ &= -\partial_x\left(\frac{1}{y}\delta\left(\frac{y}{x}\right)\right).\end{aligned} \tag{70}$$

Then upon differentiating both sides of $(x-y)\delta(y/x) = 0$ with respect to x and multiplying the result by $(x-y)$, we find from (70) that

$$(x-y)^2[\varphi_a(x), \varphi_a(y)] = -y^{-1}(x-y)^2\partial_x\delta(y/x) = 0. \tag{71}$$

Hence, condition $4'$a with $N = 2$ of theorem 10 is satisfied. Meanwhile, condition $4'$b of theorem 10 says that we must find some $T \in \operatorname{End}\mathsf{S}$ such that

$$[T, \varphi_a(x)] = \partial\varphi_a(x) \qquad \Longleftrightarrow \qquad [T, a_k] = -ka_{k-1}, \qquad T(1) = 0. \tag{72}$$

For this, we choose (below, the a_m stand for $\rho(a_m) \in \operatorname{End}\mathsf{S}$, so their multiplication is really composition)

$$T = \sum_{m>0} a_{-m-1}a_m. \tag{73}$$

Although the sum is infinite, T is still an endomorphism of S because any polynomial $v \in \mathsf{S}$ is annihilated by a_m for m sufficiently large. Furthermore, we have

$$[a_{-m-1}a_m, a_k] = -(m+1)\delta_{k-m-1}a_m, \qquad \Longrightarrow \qquad [T, a_k] = \sum_{m>0} -(m+1)\delta_{k-m-1,0}a_m = -ka_{k-1}, \tag{74}$$

in addition to the obvious property $T(1) = 0$. Thus, with this choice (73) of T, condition $4'$b of theorem 10 is satisfied. Finally, the basis (35) of condition $5'$ of theorem 10 obviously matches the basis β (67) for S. Because all of the conditions of theorem 10 are satisfied by choices (68), (69), $\mathsf{I} = \{a\}$, and (73), we conclude that $(\mathsf{S}, 1, Y)$ with

$$Y(t_{m_1}t_{m_2}\cdots t_{m_n}, x) = Y(a_{-m_1}a_{-m_2}\cdots a_{-m_n}.1, x) \tag{75}$$

$$:= :D^{m_1-1}\varphi_a(x)D^{m_2-1}\varphi_a(x)\cdots D^{m_n-1}\varphi_a(x):, \qquad \varphi_a(x) := \sum\nolimits_k a_k x^{-k-1}, \tag{76}$$

$$Y(1,x) := \mathrm{id}_{\mathsf{S}} \tag{77}$$

is a vertex algebra, and $\mathcal{D} = T$. In fact, the collection $(\mathsf{S}, 1, Y, \omega)$ is a vertex operator algebra, with respect to the grading

$$\mathsf{S} = \bigoplus_{n\geq 0} \mathsf{S}_n, \qquad \mathsf{S}_n = \{v \in \mathsf{S} \,|\, v \text{ homogeneous with } \deg(v) = n\} \qquad \deg(t_{m_1}t_{m_2}\cdots t_{m_n}) := \sum_{k=1}^{n} m_k, \tag{78}$$

where the "conformal vector" ω is any one of

$$\omega_\lambda = \frac{1}{2}t_1^2 + \lambda t_2, \qquad \lambda \in \mathbb{C}. \tag{79}$$

D. The Virasoro vertex algebra

In this section, we consider the Virasoro algebra Vir. This is the infinite dimensional complex Lie algebra with basis $\{L_k \,|\, k \in \mathbb{Z}\} \cup \{Z\}$ and commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}m(m^2-1)\delta_{n+m,0}Z, \qquad [Z, L_m] = 0, \qquad m,n \in \mathbb{Z}. \tag{80}$$

Now we consider a vector space V with basis

$$\beta = \left\{v_{m_1,m_2,\ldots,m_k} \,\middle|\, k, m_j \in \mathbb{Z}^+, m_1 \geq m_2 \geq \ldots \geq m_k \geq 2\right\} \cup \{\mathbf{1}\}. \tag{81}$$

This vector space becomes a Vir-module with *central charge* $c \in \mathbb{C}$ under the representation $\rho = \rho_c : \mathsf{Vir} \longrightarrow \mathrm{End}\,\mathsf{V}$ given by the linear extension of

$$\rho(L_n)\mathbf{1} = 0 \text{ for } n \geq -1, \qquad \rho(L_0)v_{m_1,m_2,\ldots,m_k} = \left(\sum\nolimits_{j=1}^{k} m_j\right)v_{m_1,m_2,\ldots,m_k} \tag{82}$$

$$\rho(L_{-n})\mathbf{1} = v_n \text{ for } n \geq 2, \qquad \rho(L_{-n})v_{m_1,m_2,\ldots,m_k} = v_{n,m_1,m_2,\ldots,m_k} \text{ for } n \geq m_1, \qquad \rho(Z)v = cv. \tag{83}$$

To determine the action of L_n on $v_{m_1,m_2,\ldots,m_k}$ for $n < m_1$, we apply the commutation relations (80). From this, we can show, for example, that

$$L_0.L_n.v_{m_1,m_2,\ldots,m_k} = \left(\sum\nolimits_{j=1}^{k} m_k - n\right)L_n.v_{m_1,m_2,\ldots,m_k}, \qquad \sum\nolimits_{j=1}^{k} m_k - n \geq 0, \tag{84}$$

$$L_n.v_{m_1,m_2,\ldots,m_k} = 0, \qquad \sum\nolimits_{j=1}^{k} m_k - n < 0. \tag{85}$$

Before continuing, it is worthwhile to check that (82, 83) does in fact define a representation of Vir. In fact, there is a straightforward way to do this. We let U denote the universal enveloping algebra of Vir, and we write

$$f : \mathsf{Vir} \longrightarrow \mathsf{U}, \qquad f(L_k) = \mathcal{L}_k, \qquad f(Z) = \mathcal{Z}, \tag{86}$$

where f is the canonical injection of Vir into U such that Vir is isomorphic to the Lie algebra $f(\mathsf{Vir})$, the latter having bracket $[\mathcal{L}_m, \mathcal{L}_n] := \mathcal{L}_m\mathcal{L}_n - \mathcal{L}_n\mathcal{L}_m$. Then the Poincare-Birkhoff-Witt theorem says that the set

$$\left\{\cdots \mathcal{L}_{-k}^{p_{-k}} \cdots \mathcal{L}_{-2}^{p_{-2}}\mathcal{L}_{-1}^{p_{-1}}\mathcal{Z}^p\mathcal{L}_0^{p_0}\mathcal{L}_1^{p_1}\cdots\mathcal{L}_k^{p_k}\cdots \,\middle|\, p_i \in \mathbb{Z}^+ \cup \{0\}, \text{ and all but finitely many } p_i \text{ equal zero}\right\} \tag{87}$$

is a basis for U. We note that the monomial in this basis with $p_k = 0$ for all $k \in \mathbb{Z}^+$ and $p = 0$ serves as the unit 1 in U. Next, we define the representation

$$r : \mathsf{Vir} \longrightarrow \mathrm{End}\,\mathsf{U}, \qquad r(a)u = f(a)u, \tag{88}$$

and we let $\mathsf{W} \subset \mathsf{U}$ be the left ideal generated by $\{\mathcal{Z} - c1, \mathcal{L}_k \,|\, k \geq -1\}$. Then because $r(a)w = f(a)w \in \mathsf{W}$ for all $a \in \mathsf{Vir}$ and $w \in \mathsf{W}$, we conclude that W is a submodule of U. As such, we may consider the quotient module/representation

$$\mathsf{V} := \mathsf{U}/\mathsf{W} \qquad \rho : \mathsf{Vir} \longrightarrow \mathrm{End}\,\mathsf{V}, \qquad \rho(a)[u] := [r(a)u] = [f(a)u], \qquad u \in \mathsf{U}, \qquad [u] \in \mathsf{V}. \tag{89}$$

Then with $v_{m_1,m_2,\ldots,m_k} := [\mathcal{L}_{-m_1}\mathcal{L}_{-m_2}\cdots\mathcal{L}_{-m_k}]$ for $m_1 \geq m_2 \geq \ldots \geq m_k \geq 2$ and $\mathbf{1} = [1]$, β (81) is clearly a basis for V, and the Vir-action on V given by ρ, matches (82, 82). Indeed,

$$\rho(L_n)\mathbf{1} = [\mathcal{L}_n 1] = [\mathcal{L}_n] = 0, \qquad n \geq -1, \tag{90}$$

$$\rho(L_n)\mathbf{1} = [\mathcal{L}_{-n} 1] = [\mathcal{L}_{-n}] = v_n, \qquad n \geq 2, \tag{91}$$

$$\begin{aligned}\rho(L_0)v_{m_1,m_2,\ldots,m_k} &= [\mathcal{L}_0\mathcal{L}_{-m_1}\mathcal{L}_{-m_2}\cdots\mathcal{L}_{-m_k}] \\ &= \Big(\sum\nolimits_{j=1}^{k} m_j\Big)[\mathcal{L}_0\mathcal{L}_{-m_1}\mathcal{L}_{-m_2}\cdots\mathcal{L}_{-m_k}] + [\mathcal{L}_{-m_1}\mathcal{L}_{-m_2}\cdots\mathcal{L}_{-m_k}\mathcal{L}_0] \\ &= \Big(\sum\nolimits_{j=1}^{k} m_j\Big)v_{m_1,m_2,\ldots,m_k} + 0 = \Big(\sum\nolimits_{j=1}^{k} m_j\Big)v_{m_1,m_2,\ldots,m_k},\end{aligned} \tag{92}$$

$$\rho(L_{-n})v_{m_1,m_2,\ldots,m_k} = [\mathcal{L}_{-n}\mathcal{L}_{-m_1}\mathcal{L}_{-m_2}\cdots\mathcal{L}_{-m_k}] = v_{n,m_1,m_2,\ldots,m_k}, \qquad n \geq m_1, \tag{93}$$

$$\begin{aligned}\rho(Z)v_{m_1,m_2,\ldots,m_k} &= [\mathcal{Z}\mathcal{L}_{-n}\mathcal{L}_{-m_1}\mathcal{L}_{-m_2}\cdots\mathcal{L}_{-m_k}] \\ &= [(\mathcal{Z}-c1)\mathcal{L}_{-n}\mathcal{L}_{-m_1}\mathcal{L}_{-m_2}\cdots\mathcal{L}_{-m_k}] + [c1\,\mathcal{L}_{-n}\mathcal{L}_{-m_1}\mathcal{L}_{-m_2}\cdots\mathcal{L}_{-m_k}] \\ &= [\mathcal{L}_{-n}\mathcal{L}_{-m_1}\mathcal{L}_{-m_2}\cdots\mathcal{L}_{-m_k}(\mathcal{Z}-c1)] + c\,[\mathcal{L}_{-n}\mathcal{L}_{-m_1}\mathcal{L}_{-m_2}\cdots\mathcal{L}_{-m_k}] \\ &= 0 + cv_{m_1,m_2,\ldots,m_k} = cv_{m_1,m_2,\ldots,m_k}.\end{aligned} \tag{94}$$

(To derive (92, 94), we use the commutation relations $[\mathcal{L}_0, \mathcal{L}_{-n}] = n\mathcal{L}_{-n}$ and $[\mathcal{L}_n, \mathcal{Z}] = 0$, following from (80) with $(L_n, L_m, Z) \mapsto (\mathcal{L}_n, \mathcal{L}_m, \mathcal{Z})$.) We observe that (90–94) matches (82, 83), so the former indeed defines a representation.

Now, in order to endow V with a vertex algebra structure, we must identify a vacuum vector and vertex operator $Y(v, x)$ for each $v \in \mathsf{V}$. As the notation suggests, $\mathbf{1}$ of (81) is the correct choice of vacuum vector. In order to use the reconstruction theorem 12, we must also identify a linearly independent set $\mathsf{I} \subset \mathsf{V}$ and a $T \in \operatorname{End}\mathsf{V}$ satisfying certain conditions. To begin, we suppose that there is a particular vector $a \in \mathsf{V}$ such that (34)

$$\varphi_a(x) := \sum_k L_{k-1}x^{-k-1} \in (\operatorname{End}\mathsf{V})[[x^{\pm 1}]] \qquad \Longrightarrow \qquad a_k = L_{k-1}. \tag{95}$$

Condition $3'$ requires that $\lim_{x\to 0}\varphi_a(x)\mathbf{1} = a$. This amounts to the condition $L_{k-1}\mathbf{1} = 0$ for $k \geq 0$, already satisfied in (82), and $a = L_{-2}\mathbf{1} = v_2$ (83). As such, we set $\mathsf{I} = \{a\}$. Then because any $v \in \mathsf{V}$ is annihilated by L_n for sufficiently large $n \in \mathbb{Z}^+$ thanks to (85), $\varphi_a(x)$ (95) is a field, so condition $1'$ of theorem 10 is satisfied. Also, (95) shows that condition $3'$ of this theorem is satisfied too. Next, we note that

$$\begin{aligned}[\varphi_a(x), \varphi_a(y)] &= \sum_{k,l}[L_{k-1}, L_{l-1}]x^{-k-1}y^{-l-1} \\ &= \sum_{k,l}(k-l)L_{k+l-2}x^{-k-1}y^{-l-1} + \frac{Z}{12}\sum_{k,l}(k-1)(k^2-2k)\delta_{k+l-2,0}x^{-k-1}y^{-l-1} \\ &= 2\sum_{k,m}kL_{m-1}y^{-m-1}x^{-k-1}y^{k-1} + \sum_{k,m}(-m-1)L_{m-1}y^{-m-2}x^{-k-1}y^{k} \qquad (m = k+l-1) \\ &\qquad + \frac{Z}{12}\sum_k k(k-1)(k-2)x^{-k-1}y^{k-3},\end{aligned} \tag{96}$$

Thus, we may write (96) as

$$[\varphi_a(x), \varphi_a(y)] = \frac{2}{x}\partial_y\delta\left(\frac{y}{x}\right)\varphi_a(y) + \frac{1}{x}\delta\left(\frac{y}{x}\right)\partial\varphi_a(y) + \frac{Z}{12}\partial_y^3\left(\frac{1}{x}\delta\left(\frac{y}{x}\right)\right). \tag{97}$$

Because the delta function $\delta(y/x)$ and its derivatives are annihilated by left multiplication by $(x-y)^N$ for sufficiently large N, it follows from (97) that condition $4'$a is satisfied. Next, condition $4'$b of theorem 10 says that we must find some $T \in \operatorname{End}\mathsf{S}$ such that

$$[T, \varphi_a(x)] = \partial\varphi_a(x) \qquad \Longleftrightarrow \qquad [T, L_{k-1}] = -kL_{k-2}, \qquad T(1) = 0. \tag{98}$$

For this, we choose

$$T = L_{-1}. \tag{99}$$

Thus, with this choice (73) of T, condition 4′b of theorem 10 is satisfied. Finally, the set (35) of condition 5″ of theorem 12 obviously spans V. Because all of the conditions of theorem 12 are satisfied by choices $\mathsf{I} = \{a\}$, and (99), we conclude that $(\mathsf{V}, \mathbf{1}, Y)$ with

$$Y(v_{m_1,m_2,\ldots,m_k}, x) = Y(L_{-m_1}, L_{-m_2}, \ldots, L_{-m_k}.\mathbf{1}, x) \tag{100}$$

$$:= :D^{m_1-2}\varphi_a(x) D^{m_2-2}\varphi_a(x) \cdots D^{m_n-2}\varphi_a(x):, \qquad \varphi_a(x) := \sum\nolimits_k L_{k-1}x^{-k-1}, \tag{101}$$

$$Y(\mathbf{1}, x) := \mathrm{id}_{\mathsf{V}} \tag{102}$$

is a vertex algebra, and $\mathcal{D} = T$. (If $c = 1$, then $\varphi_a(x)$ may be regarded as the stress-energy tensor of the two-dimensional massless free boson in CFT.) Furthermore, the collection $(\mathsf{V}, \mathbf{1}, Y, \omega)$ is a vertex operator algebra, with respect to the grading

$$\mathsf{V} = \bigoplus_{n \geq 0} \mathsf{V}_n, \qquad \mathsf{V}_n = \{v \in \mathsf{V} \mid L_0.v = nv\}, \tag{103}$$

where the "conformal vector" ω is

$$\omega = L_{-2}\mathbf{1}. \tag{104}$$

Indeed, we have

$$Y(\omega, x) = Y(L_{-2}.\mathbf{1}.x) = \varphi_a(x) = \sum\nolimits_k L_k x^{-k-2}, \tag{105}$$

and from this with (80), it is completely evident that ω satisfies the conditions of a conformal vector (with central charge c given by $\rho(Z) = c\,\mathrm{id}_{\mathsf{V}}$). Furthermore, by theorem 10 and proposition 3.1.21 of [1], we have

$$L_{-1} = T = \mathcal{D} \qquad \Longrightarrow \qquad Y(L_{-1}.v, x) = \partial Y(v, x). \tag{106}$$

From these facts, we conclude that $(\mathsf{V}, \mathbf{1}, Y, \omega)$ is indeed a vertex operator algebra.

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[1] J. Lepowski and H. Li, *Introduction to Vertex Operator Algebras and their Representations.*
[2] M. Schottenloher, *A Mathematical Introduction to Conformal Field Theory.*
[3] E. Frenkel, *Langlands Correspondence for Loop Groups.*
[4] A. Weekes, *Talk 3: Operator Product Expansions.* http://www.math.toronto.edu/aweekes/OPE.pdf.
[5] E. Frenkel, D. Ben-Zvi, *Vertex Algebras and Algebraic Curves: Second Edition.*
[6] P. Goddard, *Meromorphic conformal field theory.* In *Infinite Dimensional Lie Algebras and Groups: Proceedings of the Conference*, ed. V. Kac.