TODAY
- more axiomatic basics
- JACOBI IDENTITY $\Longleftrightarrow$ $\vdots$
- def of VOA

Quick review

V vect. sp.

$\mathrm{End}(V) = \{ \text{linear maps } V \to V \}$

$V[[x, x^{-1}]] = \{ \text{formal series } \sum_{n \in \mathbb{Z}} v_n \cdot x^n \}$

(ex. $V = \mathbb{C}$, $\delta(x) = \sum_{n \in \mathbb{Z}} x^n$)

binomial expansion convention $(x+y)^n = \sum_{k \geq 0} \binom{n}{k} x^{n-k} y^k$

$$\delta(x) = (1-x)^{-1} + (x-1)^{-1}$$

note: ① $(1-x) \cdot \delta(x) = 0$ ("can't divide by $(1-x)$")

② $\frac{1}{n!} \frac{d^n}{dx^n} \delta(x) = (1-x)^{-1-n} - (-x+1)^{-n-1}$

vertex op. $Y(\cdot, x) : V \to \left(\mathrm{End}(V)\right)[[x, x^{-1}]]$

$v \longmapsto Y(v, x) = \sum_{n \in \mathbb{Z}} x^{-1-n} \cdot v_n$

convention of labeling the coefs

Jacobi identity

$$x_0^{-1} \delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u, x_1) Y(v, x_2) - x_0^{-1} \delta\left(\frac{-x_2 + x_1}{x_0}\right) Y(v, x_1) Y(u, x_2)$$
$$= x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_2}\right) Y\left(Y(u, x_0) v, x_2\right)$$

Compare with Jacobi identity in Lie algebra $\mathfrak{g}$:

$U, V \in \mathfrak{g}$ $\qquad \mathrm{ad}_U(V) = [U, V]$

Jacobi : $\mathrm{ad}_U \, \mathrm{ad}_V - \mathrm{ad}_V \, \mathrm{ad}_U = \mathrm{ad}_{\mathrm{ad}_U(V)}$

Consequences and equivalent axioms

① Res_{x_0} (JACOBI) :

$$[Y(u,x_1), Y(v,x_2)] = \mathrm{Res}_{x_0} \text{ (right hand side)}$$

"commutator formula"

② "Associator formula"

$$Y(Y(u,x_0)v, x_2) - Y(u, x_0+x_2)Y(v,x_2)$$
$$= \mathrm{Res}_{x_1}\left(x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right) Y(v,x_2)Y(u,x_1)\right)$$

Proposition Given $u,v \in V$ $\exists K$ such that for $k \geq K$

$$(x_1-x_2)^k [Y(u,x_1), Y(v,x_2)] = 0$$

("Weak commutativity")

Proof: Multiply JACOBI by x_0^k, $k \geq 0$. Take Res_{x_0}.

$$\text{LHS} = (x_1-x_2)^k [Y(u,x_1), Y(v,x_2)]$$

$$\text{RHS} = \mathrm{Res}_{x_0} \underbrace{x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)}_{\sum_{\ell \geq 0} x_0^\ell (\dots)} x_0^k \, Y(\underbrace{Y(u,x_0)v}_{\text{has finitely many negative powers by lower truncation property.}}, x_2)$$

by binomial expansion conv.

□

D-derivative property

$D : V \to V$ defined by $D(v) = v_{-2}\mathbb{1}$

Proposition $Y(Dv, x) = \frac{d}{dx} Y(v,x)$

Proof Homework.

Key points:
- $Y(Dv,x) = Y(v_{-2}\mathbb{1}, x) = $ coef of $Y(Y(\cdot)\mathbb{1}, x)$
- $\frac{1}{n!}\left(\frac{\partial}{\partial x_2}\right)^n x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) = (x_1-x_2)^{-n-1} - (-x_2+x_1)^{-n-1}$
- JACOBI $\Rightarrow$ $Y(Y(u,x_0)v, x_2)$
$$= \mathrm{Res}_{x_1}\left(x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_1Y_2 - x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y_2Y_1\right)$$

□

Corollary ("D-derivative property")

$$Y(e^{x_0 D} v, x) = e^{x_0 \frac{d}{dx}} Y(v, x)$$

Theorem In the def of VA, the Jacobi identity can be replaced by weak commutativity and D-derivative property.

Proposition (Skew symmetry)

$$Y(u,x)v = e^{xD} Y(v,-x)u$$

Thm In the VA axioms, JACOBI can be replaced by skew symmetry and weak associativity.

... and other equivalent axioms ...

Def (Vertex Operator Algebra)

- V $\mathbb{Z}$-graded vector space, $V = \bigoplus_{n \in \mathbb{Z}} V(n)$
  with $\dim(V(n)) < \infty$ $\forall n$
  and $V(n) = \{0\}$ for n sufficiently negative.

(Terminology: $v \in V(n)$, $wt(v) = n$ "weight")

- $(V, Y, \mathbb{1})$ is a Vertex Algebra
- $\exists\, \omega \in V(2)$ "conformal vector" s.t.

  (A) define $L(n) \in End(V)$ by $Y(\omega, x) = \sum_{n \in \mathbb{Z}} L(n) x^{-2-n}$
  $$= \sum_{n \in \mathbb{Z}} \omega_n x^{-1-n}$$

  Then
  $$[L(n), L(m)] = (m-n) L(m+n) + \frac{1}{12}(m^3 - m)\, \delta_{m+n,0}\, c_V \cdot id_V$$
  for some $c_V \in \mathbb{C}$

  i.e. $(L(n))$ defines a rep. of Virasoro algebra on V.

⋮

⋮

(B) If $v \in V(n)$ then

$L(0)v = n\,v = wt(v)\cdot v$

(i.e. grading by $L(0)$ eigenspaces (eigenvalues))

(C) $Y(L(-1)v, x) = \frac{d}{dx} Y(v,x)$

"$L(-1)$ - derivative property"

<u>Remarks about ω</u>

① Why is it called the "conformal vector"?

Work in $\mathbb{C}[x,x^{-1}]$.

Formal derivation $p(x)\frac{d}{dx}$, $p(x) \in \mathbb{C}[x,x^{-1}]$.

FACT: Every derivation of the algebra $\mathbb{C}[x,x^{-1}]$ are of this form

$$\mathrm{Der}(\mathbb{C}[x,x^{-1}]) = \left\{ p(x)\frac{d}{dx} \;\middle|\; p(x) \in \mathbb{C}[x,x^{-1}] \right\}$$

" Lie algebra of derivations has basis $d_n = -x^{n+1}\frac{d}{dx}$

$$[d_n, d_m] = (n-m)\, d_{n+m}$$

Virasoro alg = central extension of $\mathrm{Der}(\mathbb{C}[x,x^{-1}])$.

BUT: Derivations are infinitesimal generators of conformal transformations.

e.g. $e^{y\frac{d}{dx}} f(x) = f(x+y)$

in general $e^{t\cdot x^{n+1}\frac{d}{dx}} = ???$

② Commutator formula

⇓

$$[L(-1), Y(u,x)] = Y(L(-1)u, x) = \frac{d}{dx} Y(u,x)$$

$$[L(0), Y(u,x)] = \ldots$$

③ Recall in the def of VA

$Y(v,x)\mathbb{1} \in V[[x]]$ and $\lim_{x\to 0} Y(v,x)\mathbb{1} = v$.

Put $v = \omega$. $Y(\omega,x)\mathbb{1} = \sum_{n\in\mathbb{Z}} x^{-2-n} L(n)\mathbb{1} = x^0\omega + \mathcal{O}(x)$

i.e. $L(-1)\mathbb{1} = 0$, $L(0)\mathbb{1} = 0$, $L(n)\mathbb{1} = 0$ $\forall n>0$

and $L(-2)\mathbb{1} = \omega$.

④ One can prove $D = L(-1) \in \mathrm{End}(V)$
(Looks like the derivative property together with creation property should uniquely determine the endomorphism ...)

Rationality results

Summary: Define the restricted dual space
$$V' = \bigoplus_{n\in\mathbb{Z}} V_{(n)}^*$$

Thm
$\langle w', Y(u,x_1)Y(v,x_2)w\rangle \in \mathbb{C}[x_1,x_1^{-1},x_2,x_2^{-1}]_S$.
$\forall$ $u,v,w\in V$, $w'\in V'$

(means "rational functions expanded in a particular way")

$\langle w', Y_1Y_2w\rangle$ and $\langle w', Y_2Y_1w\rangle$ are "convergent" and two different expansions of the same rational function.