CFT, VOA, ETC

Today Lepowsky & Li book Ch 1-3 definitions

Def (Vertex Algebra)

- data:
  - V vect. sp.
  - linear map $V \otimes V \to V[[x, x^{-1}]]$
    (equivalently $Y(\cdot, x) : V \to (\mathrm{End}\, V)[[x, x^{-1}]]$
    $v \mapsto Y(v,x) = \sum_{n \in \mathbb{Z}} v_n x^{-1-n}$
    $v_n \in \mathrm{End}\, V$)
  - distinguished vector $\mathbb{1} \in V$ (vacuum)

- axioms $u, v \in V$

(a) $u_n v = 0$ for n sufficiently large
(lower truncation condition)
($\Leftrightarrow$ $Y(u,x)v \in V((x))$)

(b) $Y(\mathbb{1}, x) = x^0 \cdot \mathrm{id}_V$

(c) $Y(v,x)\mathbb{1} \in V[[x]]$ and $\lim_{x \to 0} Y(v,x)\mathbb{1} = v$
(state-field correspondence / creation property)

(d) "JACOBI IDENTITY"

$$x_0^{-1} \delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u, x_1) Y(v, x_2)$$
$$- x_0^{-1} \delta\left(\frac{x_2 - x_1}{-x_0}\right) Y(v, x_2) Y(u, x_1)$$
$$= x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_2}\right) Y\left(Y(u, x_0) v, x_2\right)$$

Formal calculus

V vect sp.

$V[[x, x^{-1}]]$ vect. sp. of formal doubly infinite series

$\shortparallel$

$\left\{ \sum_{n\in\mathbb{Z}} v_n x^n \,\middle|\, v_n \in V \right\}$

$$V((x)) = \left\{ \sum_{n\in\mathbb{Z}} v_n x^n \,\middle|\, v_n \in V \text{ and } v_n = 0 \text{ for } n << 0 \right\}$$

Define operations when individual coefficients (of x^k) are finite sums.

Formal delta-function

$$\delta(x) = \sum_{n\in\mathbb{Z}} x^n$$

$$\left(\begin{array}{ll} \delta(x) f(x) = f(1)\,\delta(x) & f(x) \in V[x, x^{-1}] \\ \delta(x)\, x^n \overset{\uparrow}{=} \delta(x) & \end{array} \right)$$

Formal limit $\lim_{x_1 \to x_2} (\dots) = (\dots)\big|_{\text{substitute } x_1 = x_2}$

Define when coefs are finite sums.

Two variable delta-function

$f(x_1, x_2) \in V[[x_1, x_1^{-1}, x_2, x_2^{-1}]]$ s.t. $\lim_{x_1 \to x_2} f(x_1, x_2)\ \exists$

Then $f(x_1, x_2)\,\delta\!\left(\frac{x_1}{x_2}\right) = f(x_1, x_1)\,\delta\!\left(\frac{x_1}{x_2}\right) = f(x_2, x_2)\,\delta\!\left(\frac{x_1}{x_2}\right)$

Residues $\mathrm{Res}_x \left(\sum_{n\in\mathbb{Z}} v_n x^n \right) = v_{-1}$

Ex $\mathrm{Res}_x\, \delta(x) f(x) = \mathrm{Res}_x\, \delta(x) \cdot f(1) = f(1)$

Binomial expansion convention

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \quad \text{if} \quad n \in \mathbb{N}$$

BUT generally: $$(x+y)^n = \sum_{k\in\mathbb{N}} \binom{n}{k} x^{n-k} y^k$$

THIS IS A CHOICE WE MAKE (DEFINITION)

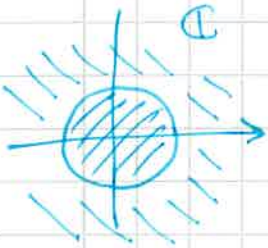
always expand in positive powers of the second variable

i.e. in region $|y| < |x|$

Example/warning: $(x+y)^n \neq (y+x)^n$ for $n<0$.

Example: $\delta(x) = (1-x)^{-1} + (x-1)^{-1}$ "expansion of zero"

Note: In $\mathbb{C}$



$\frac{1}{1-z}$ in $|z|<1$: $\frac{1}{1-z} = \sum_{n\geq 0} z^n$

in $|z|>1$: $\frac{1}{z-1} = \frac{1}{z(1-1/z)} = z^{-1}\sum_{n\geq 0} z^{-n}$

Formal derivative: $\frac{d}{dx} \sum_n v_n x^n = \sum_n n v_n \cdot x^{n-1}$

Formal Taylor expansion

$$e^{y\frac{d}{dx}} v(x) = v(x+y)$$

proof: $$e^{y\frac{d}{dx}} v(x) = \sum_{n\in\mathbb{Z}} \sum_{j\geq 0} \frac{y^j}{j!} \left(\frac{d}{dx}\right)^j v_n x^n$$

$$= \sum_{n\in\mathbb{Z}} \sum_{j\geq 0} \frac{n!}{(n-j)!\,j!} y^j v_n x^{n-j} \qquad \left(\sum_j \cdots = (x+y)^n\right)$$

$$= \sum_{n\in\mathbb{Z}} v_n \cdot (x+y)^n =: v(x+y)$$

Delta function relations

(help with understanding the Jacobi identity)

series $$x_1^{-1}\,\delta\!\left(\frac{x_2+x_0}{x_1}\right) = \sum_{n\in\mathbb{Z}}\ \sum_{j\geq 0} \binom{n}{j}\, x_1^{-n-1}\, x_2^{n-j}\, x_0^{j}$$

has only positive powers of x_0

Two identities:

(A) : $$x_2^{-1}\,\delta\!\left(\frac{x_1-x_0}{x_2}\right) = x_1^{-1}\,\delta\!\left(\frac{x_2+x_0}{x_1}\right)$$

(B) : $$x_0^{-1}\,\delta\!\left(\frac{x_1-x_2}{x_0}\right) - x_0^{-1}\,\delta\!\left(\frac{-x_2+x_1}{x_0}\right) = x_2^{-1}\,\delta\!\left(\frac{x_1-x_0}{x_2}\right)$$

Look at the relation to residues:

Recall $$\mathrm{Res}_x\, \delta(x) f(x) = f(1)$$

think of this as a map $f(x) \longmapsto f(1)$

In complex analysis also

$$f(1) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-1}\,dz$$ (integral over contour around 1)

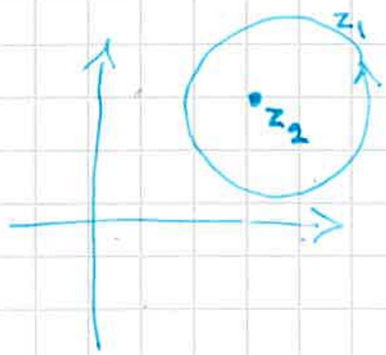
Two variable case

$x_2^{-1}\,\delta\!\left(\frac{x_1}{x_2}\right)$ residues and $\oint$?

$$\mathrm{Res}_{x_1}\, x_2^{-1}\,\delta\!\left(\frac{x_1}{x_2}\right) f(x_1) = f(x_2)$$

Compare to

$$f(z_2) = \frac{1}{2\pi i}\oint_{\text{around } z_2} \frac{f(z_1)}{z_1-z_2}\,dz_1$$



Talk about VA axioms:

vaguely:

- $Y(v,x)u$ can be seen as infinitely many product operations $V \otimes V \to V$

  $v_n u$ $\quad (v \in V,\ v_n \in \mathrm{End}(V) \quad v_n u = v_n(u) \in V)$

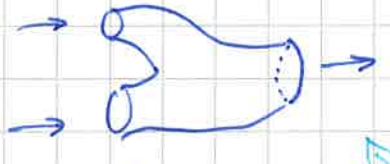
TQFT

$\rightsquigarrow$ PRODUCT

only one such (topological) space

versus

CFT



$\rightsquigarrow$ INFINITELY MANY PRODUCTS

(∞-dim. moduli space)

Conformal equivalence classes of Riemann surfaces with boundary (not just topological!)

note: $Y(v,x) = 0 \iff v = 0$ (by creation property)

$\therefore$ $Y(\cdot, x)$ is injective

$\therefore$ can transport the VA structure to $\{ Y(v,x) \mid v \in V \} \subset (\mathrm{End}\, V)[[x, x^{-1}]]$

"state" $v \in V$ $\qquad$ "operator"

this is the "state / field correspondence"