SCHIFFER VARIATION IN TEICHMÜLLER SPACE, DETERMINANT LINE BUNDLES AND MODULAR FUNCTORS

BY DAVID AUBREY ALFRED RADNELL

A dissertation submitted to the

Graduate School—New Brunswick

Rutgers, The State University of New Jersey

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

Graduate Program in Mathematics

Written under the direction of

Yi-Zhi Huang

and approved by

New Brunswick, New Jersey

October, 2003

ABSTRACT OF THE DISSERTATION

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Dissertation Director: Yi-Zhi Huang

The pure mathematical incarnation of conformal field theory was introduced by Segal [48] and Kontsevich around 1987. Recently, Hu and Kriz [26] further rigorized Segal's definition. Conformal field theory is intimately connected to vertex operator algebras and the complex geometry of Riemann surfaces with analytically parametrized boundaries. This thesis is centered on the analytic and geometric aspects of this theory.

An explicit description of the complex structure of the infinite-dimensional moduli space of Riemann surfaces with analytically parametrized boundary components is given and the holomorphicity of the sewing operation is proved. The determinant line bundle is shown to be a holomorphic bundle over this moduli space and the sewing operation is proved to be holomorphic on these bundles. Applications to modular functors, which are high-rank generalizations of the determinant line bundle, are discussed.

All these results are needed in order to have rigorous definitions of a holomorphic (or chiral) conformal field theory and a holomorphic modular functor. So certainly this is a necessary step in the on-going project to construct higher-genus conformal field theories from vertex operator algebras.

The formulation and proofs of these results rely on deep aspects of analytic Teichmüller theory and quasiconformal mappings, the uniformization of higher-genus Riemann surfaces, and Schiffer variation. To my knowledge these techniques have not previously been applied in this context and will have continued applicability.

Acknowledgements

This thesis began with some problems Yi-Zhi Huang suggested to me several years ago. Throughout the long process of understanding and then solving these problems he has been of constant help and support. His deep insight has always been an inspiration and a guide. But more than this, Yi-Zhi Huang has been a mentor in the true sense of the word. For all of this I offer my sincerest thanks. I also thank him for considerable help with Chapter 6.

Many people contributed directly to this thesis. I thank Feng Luo for many helpful discussion and for telling me the proof of Lemma 5.5.6. He also inspired me to try and draw good pictures. My gratitude goes to Alexander Kirillov Jr. for pointing out some errors in a draft of this thesis. Jim Lepowsky has been of constant encouragement and support throughout my graduate studies and I express my deep appreciation for that.

Schiffer variation is one of the fundamental tools used in this thesis. The idea due to Frederick Gardiner, of interpreting Schiffer variation as a quasiconformal deformation of complex structure, inspired me to interpret the sewing operation in a similar way. Certainly without his result this thesis would have been entirely different. I thank him for that as well as for discussions. I must mention the excellent book of the late Subhashis Nag where I first came across these results.

I would also like to thank my fellow graduate students, David Galvin, Alfredo Rios, James Taylor, Yuka Taylor, Jason Tedor, Lee Zhao and many others who have all contributed to this thesis in some way.

During my six years at Rutgers I benefited mathematically, as both a researcher and a teacher, from the many excellent courses I attended. In this regard I would especially like to thank Jim Lepowsky for teaching with such conciseness, clarity and insight. As the graduate director for much of my time, Stephen Greenfield must be thanked for doing an excellent job and helping in all aspects of teaching and research. I thank Sheldon Goldstein and Michael Kiessling for many interesting conversations on mathematics and physics. In particular I am indebted to Sheldon Goldstein and James Taylor for teaching me Bohmian mechanics.

On a more personal note I will mention David Brander, Maxime Dion, Ivan Ermanoski, David Galvin, Regina and Andrew Grenfell, Masud Haque, Antun Milas, Francisco Ojeda, Indranil Paul, Cliff Smyth, Jason Tedor and Lee Zhao for sharing many adventures and being great company over the years.

Life certainly would not have been so good without the company of Alfredo and Beatriz, who are two of the kindest and funniest people I know. My gratitude also goes to my parents whose love, support and zest for life and learning has no doubt influenced me greatly.

And lastly, which is the most important place, I mention my wife Laura. I know not the words to express my gratitude for all the love she has given and all the great times we have shared since we first met at the beginning of my graduate studies. And certainly without the good kick she has given me from time to time this thesis may never have been completed.

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Chapter 1

Introduction

From a technical standpoint this thesis is a study of the complex geometry of the moduli space of Riemann surfaces with analytically parametrized boundaries, the determinant line bundle and sewing operation. The context and motivation must be understood in order to fully appreciate the results and techniques, as well as their future uses. In particular this work is a necessary step in the on-going project to construct conformal field theory (CFT) from vertex operator algebras. That project and its relation to geometry is surveyed in Huang [32].

It is implicit in the definition of holomorphic weakly conformal field theory that the determinant line bundle is a holomorphic line bundle over the moduli space of Riemann surfaces with parametrized boundaries. Also, the sewing operation must be holomorphic. Surprisingly, these holomorphicity issues have never been addressed in the literature. In this thesis, these problems are are formulated rigorously and solved completely.

A brief description of conformal field theory and its mathematical definition is given in the next section. Following that is a section containing an overview of the results obtained in this thesis and the methods used.

1.1 Conformal field theory

Conformal field theory (CFT) originally arose in physics from various two-dimensional statistical mechanics models where it is used to explain critical phenomena in phase transitions. In the seminal paper of Belavin, Polyakov and Zamolodchikov [7] much of the structure of CFT was encoded in the notion of a chiral algebra, at the physical level of rigor. These algebras are essentially equivalent to vertex operator algebras which were developed independently in mathematics by Borcherds [9] and Frenkel, Lepowsky and Meurman [15].

At around the same time in string theory the study of the geometry of CFT was introduced by Friedan and Shenker [16]. In this context the two dimensional objects of study are the world sheets of strings. The world sheets of interacting strings are actually Riemann surfaces with boundary. The conformal structure arises naturally from physics.

In the path integral approach to quantum field theory, one must "sum" over all possible paths. In the case of string interactions the possible paths are the possible Riemann surfaces joining the prescribed boundaries (strings). The conformal invariance inherent in the physics means the path integrals are taken over the moduli space. However, it is well know that path integrals are not rigorous, and it is doubted whether they can in fact be made rigorous in a direct way. Segal [48] and Kontsevich extracted the mathematical properties such a theory should have and gave a purely mathematical definition of conformal field theory. Substantial work was done recently by Hu and Kriz [26] to ruther rigorize this definition.

In quantum mechanics the state space of a system is supposed to be described by a Hilbert space and operators are associated with interactions. In CFT each boundary circle must be associated with a Hilbert space and each interaction (Riemann surface) must be associated with an operator. Such Riemann surfaces can be sewn using the boundary parametrizations and certain geometric properties of this operation translate into relations of the corresponding operators.

Collecting these ideas we define, roughly speaking, a conformal field theory to be a projective linear representation of the moduli space of compact Riemann surfaces with parametrized boundaries equipped with the sewing operation. Although this definition has existed for fifteen years no general construction for arbitrary genus has been given. This attests to the richness of the mathematical structure of CFT and the difficulties faced in its construction.

The genus-zero theory has been completely worked out by Huang in, [28], [29], [30], [31], and [33]. The genus-one theory is also essentially complete due to the work of

Zhu [51] and Huang [27]. Free fermion theories were outlined by Segal [48] and have recently been elaborated on by Kriz [37]. Many people have also worked on and tried to understand the algebro-geometric and topological aspects of higher-genus theory. Some key works in this direction are [4], [5], [6],[14], [18],[34], [45], [49], and [50]. To construct CFT completely however, many *holomorphicity* issues must be addressed.

We now present a slightly more detailed sketch of the definition of a *holomorphic weakly conformal field theory* in the sense of Segal. Consider the category, $\mathcal{C}$, whose objects are ordered sets of copies of the unit circle S^1 and whose morphisms are conformal equivalence classes of Riemann surfaces with oriented, ordered, and analytically parametrized boundaries such that the negatively (positively) oriented boundaries are parametrized by the copies of S^1 in the domain (co-domain). Composition of morphisms is defined by the sewing of oppositely oriented boundary components in the unique way specified by the parametrizations. A conformal field theory is a functor from this category to the category of complete locally convex vector spaces over $\mathbb{C}$, satisfying certain natural axioms.

To capture the necessary structure of the conformal anomaly in CFT one needs to consider projective functors, or equivalently, extensions of $\mathcal{C}$ by $\mathbb{C}$. To each morphism (Riemann surface, Σ) one must assign a complex line L_Σ and a map

$$l_{\Sigma_1\Sigma_2} : L_{\Sigma_1} \otimes L_{\Sigma_2} \longrightarrow L_{\Sigma_1 \# \Sigma_2}$$

where # denotes the sewing of two surfaces. The functorial properties of a holomorphic weakly conformal field theory require that these complex lines form a holomorphic line bundle over the moduli space of Riemann surfaces with analytically parametrized boundaries and the sewing operation must be holomorphic in the sense that $l_{\Sigma_1\Sigma_2}$ depends holomorphically on Σ_1 and Σ_2. The appropriate bundle is the *determinant line bundle.* The notation of a holomorphic weakly conformal field theory also includes the possibility of replacing the determinant line bundle by high-rank generalizations which are called *modular functors.*

It turns out that serious mathematical work must be done to make the preceeding statements and constructions rigorous. The notation of CFT is relevent not just in physics, but also in mathematics, becasue many deep consequences have already been derived. Having a complete mathematical definition is therefore important, as it is a necessary step in the construction problem.

1.2 Results and methods

The main goal of this thesis is to prove the following results which are necessary in giving a rigorous definition of a *holomorphic weakly conformal field theory* and a *holomorphic modular functor.*

Theorem 1.2.1. *The infinite-dimensional moduli space of Riemann surfaces with analytically parametrized boundaries is a complex manifold and the sewing operation is holomorphic. The determinant lines form a holomorphic line bundle over this moduli space and sewing gives a holomorphic operation on these bundles.*

This statement should not be thought of as a single theorem but rather must be broken down into several pieces.

- The moduli space must first be proved to be an infinite-dimensional complex manifold and its complex structure must be given in a usable form. This is done in Chapter 3.
- In Chapter 4 the sewing operation (as an operation on these moduli spaces) is proved to be holomorphic.
- The holomorphic bundle structure of the determinant lines and holomorphicity of the sewing is finally proved in Chapter 5.

These results are non-trivial and require techniques from deep parts of Teichmüller theory, quasiconformal mappings, the uniformization theorem and Schiffer variation. The result of Hatcher and Thurston on the connectivity of the maximal multicurve complex is also needed. Even the genus-zero case is non-trivial and was first proved by Huang in [30].

Our methods and results can be applied to modular functors. The determinant line bundle over the moduli space together with the sewing operation is a special case of a *modular functor* (to be defined later). Roughly speaking, in a modular functor, the determinant lines are replaced with finite dimensional vector spaces. Chapter 6 is dedicated to the proof of the following theorem, as well as rigorizing the holomorphicity aspects of the definition of a (holomorphic) modular functor.

Theorem 1.2.2. *Consider a modular functor with a canonical projectively flat connection on the associated holomorphic vector bundle. This conection is determined by its restriction to the vector (line) bundles over the moduli spaces of disks and annuli. Moreover, these connections are determined by a central charge and a set of weights associated to central extensions of* $\mathrm{Diff}^+(S^1)$.

We now discuss the methods used to prove the series of results outlined above. A *rigged surface* is a Riemann surface Σ, of genus g, with ordered and oriented punctures $p_1, \ldots, p_n$ and analytic coordinates $\phi_1, \ldots, \phi_n$ defined in neighborhoods of the punctures. It is required that either $\phi_i(p_i) = 0$ or $\phi_i(p_i) = \infty$ depending on the orientation of the puncture p_i. The term *local coordinates* will always refer to the ϕ_i. Two rigged surfaces are called *conformally equivalent* if there is a biholomorphism between them that preserves the punctures and local coordinates. Let $\mathcal{O}$ be the complex (LB)-manifold of convergent power series in a neighborhood of $0 \in \mathbb{C}$ such that $f(0) = 0$ and $f'(0) \neq 0$. Let $T(\Sigma)$ be the Teichmüller space of the punctured surface and let $\widetilde{T}(\Sigma)$ be the *Teichmüller space of rigged surfaces* which is defined in the natural way. Let $\mathcal{M}(\Sigma)$ and $\widetilde{\mathcal{M}}(\Sigma)$ be the corresponding moduli spaces. The moduli space of Riemann surfaces with analytically parametrized boundaries is contained in $\widetilde{\mathcal{M}}(\Sigma)$. As in the general theory of Teichmüller spaces, we consider only the cases where $2g - 2 + n > 0$ (the excluded cases are covered by the genus-zero and genus-one theory).

Teichmüller theory cannot be used directly to understand $\widetilde{T}(\Sigma)$ because any marking map $f : \Sigma \to \Sigma_1$ does not preserve analyticity of the local coordinates at the punctures. So local coordinates on Σ and Σ_1 cannot be compared in this way.

Let U be the upper-half plane and G a Fuchsian group such that $\Sigma = U/G$. Let μ be a Beltrami differential, let $[\mu] \in T(G)$ and let w^μ be the normalized quasiconformal solution of the Beltrami equation on the cover U. It is well known that w^μ depends analytically on μ. Let $G^\mu = w^\mu \circ G \circ (w^\mu)^{-1}$. The key fact is that the Riemann surface $\Sigma^\mu = w^\mu(U)/G^\mu$ depends only on the equivalence class $[\mu]$. This can be thought of as a higher-genus uniformization theorem. The Bers fiber space $F(G)$, has base space $T(G)$, fibers $w^\mu(U)$ over $[\mu]$, and is a complex manifold. There is a modular group action on $F(G)$ which produces a marked n-punctured holomorphic fiber space $V(G)$ which has base space $T(G)$ and fibers Σ^μ. The space $V(G)$ is universal in the sense that any n-punctured holomorphic family of Riemann surfaces map biholomorphically into $V(G)$. See for example Nag [44]. (Note that $V(G)$ is often called the universal Teichmüller curve).

We next describe a method for producing analytic coordinates on Teichmüller called *Schiffer variation.* The ideas behind this method, as well as the result itself, play a fundamental role throughout this thesis.

Schiffer variation is performed by removing a small *disk* from the Riemann surface and filling the whole by sewing in a disk that has been deformed. The deformation parameter gives an analytic coordinate in $T(\Sigma)$. The main theorem says that variations on $\dim(T(\Sigma))$ disks gives an analytic coordinate chart on $T(\Sigma)$ and these disks can be chosen essentially arbitrarily. The analyticity is proved by producing a quasiconformal map, $\nu^\epsilon : \Sigma \to \Sigma^\epsilon$, between the original and varied surfaces that depends analytically on the parameter ϵ.

Combining these last two ideas leads to the first result of this work.

Theorem 1.2.3. *The Teichmüller space of rigged surfaces $\widetilde{T}(\Sigma)$ is a complex manifold and moreover has a bundle structure with fiber model the infinite-dimensional space $\mathcal{O}$.*

Outline of Proof. Local trivializations will be defined using Schiffer variation. The quasiconformal marking map $\nu^\epsilon : \Sigma \to \Sigma^\epsilon$ is the identity away from from the disks where the Schiffer variation is performed. By choosing these disks away from the punctures, ν^ϵ will be the identity in a neighborhood of the punctures. Thus the local coordinates on Σ^ϵ can be naturally identified with the local coordinates on Σ.

Given two such trivialization the transition function must be proved to be holomorphic. First it is shown directly that the family of surfaces Σ^ϵ produced by Schiffer variation is marked holomorphic family. The two trivializations can be compared by mapping these families into $V(G)$. The universality of $V(G)$ guarantees such maps exist and are holomorphic. □

The (pure) mapping class group acts on $\widetilde{T}(\Sigma)$ via its usual action on $T(\Sigma)$. The action is fixed-point free as any automorphism preserving the local coordinates must be the identity.

Theorem 1.2.4. *The mapping class group acts on $\widetilde{T}(\Sigma)$ with no fixed points and the quotient is the moduli space of rigged surfaces $\widetilde{\mathcal{M}}(\Sigma)$. Hence this moduli space is an infinite-dimensional complex manifold.*

The next step is prove the holomorphicity of the sewing operation. We now consider only local coordinates at the punctures whose images contain the unit disk. By deleting the preimages of these unit disks we produce a Riemann surface with analytically parametrized boundaries. Let ϕ and ψ be respectively, positively and negatively oriented parametrizations of boundary components of Σ_1 and Σ_2. These surfaces can be sewn in a canonical way by the boundary identification $\psi^{-1} \circ \phi$.

Theorem 1.2.5. *The surface obtained from sewing depends analytically on the local coordinates at the punctures. In other words, if the coordinates depend analytically on a parameter then the resultant surface varies analytically in the Teichmüller space.*

Outline of Proof. The proof was inspired by the interpretation, in Gardiner [20] (see also Nag [44]), of Schiffer variation as a quasiconformal deformation of conformal structure.

Sewing Riemann surfaces along their analytically parametrized boundaries can be understood in a manner similar to Schiffer variation. Let Σ_1 and Σ_2 be the result of cutting a surface Σ along a closed loop. Let ϕ and ψ be oppositely oriented parametrizations of the new boundaries on Σ_1 and Σ_2. A new surface $\Sigma_1 \# \Sigma_2$ is formed by sewing using these parametrizations.

Using the inclusions of Σ_1 and Σ_2 into Σ, the map $\psi \circ (\phi)^{-1}$ makes sense as an orientation-preserving map from S^1 to itself. We explicitly construct a quasiconformal map on an annulus in $\mathbb{C}$ with outer boundary S^1, such that the map is the identity on the inner circle and is $\psi \circ (\phi)^{-1}$ on S^1. One should think of this map as a quasiconformal *untwisting* of $\psi \circ (\phi)^{-1}$. This map is carefully constructed to depend analytically on ϕ and ψ.

From this we produce a quasiconformal map between Σ and $\Sigma_1 \# \Sigma_2$ that depends analytically on ϕ and ψ. The analyticity relies on some technical results of Bers [8]. □

The determinant line Det_Σ, of a Riemann surface Σ with analytically parametrized boundary is defined by the determinant of the Fredholm operator $\overline{\partial}$ with certain boundary condition. Using some deep analysis, Huang [30] proved that in genus-zero these form a holomorphic bundle over the moduli space $\widetilde{\mathcal{M}}(\Sigma)$.

We define the determinant line of an element in Teichüller space to be the determinant line of the canonical representative Σ^μ. This defines a fiber space over the rigged Teichmüller space. The basic idea is to use pants decompositions in conjunction with Schiffer variation to give local trivializations of the determinant lines.

Consider a decomposition of a surface Σ into pants (3-holed spheres) $\Sigma_1, \ldots, \Sigma_n$. We get canonical isomorphisms

$$\mathrm{Det}_\Sigma \to \mathrm{Det}_{\Sigma_1} \otimes \ldots \otimes \mathrm{Det}_{\Sigma_N} \to \mathbb{C}$$

by repeated use of the fundamental sewing isomorhpism of determinant lines and the genus-zero theory. To produce a bundle structure we need local trivializations. The problem is how to consistently choose pants decompositions on neighboring surfaces in Teichmüller space. Schiffer varition produces a neighborhood consisting of surfaces Σ^{μ_ϵ}. The marking map $f^{\mu_\epsilon} : \Sigma \to \Sigma^{\mu_\epsilon}$ is holomorphic away from the disk where Schiffer variation is performed. If we choose a pants decomposition of Σ such that none of the curves intersect that disk, then f^{μ_ϵ} induces a pants decomposition of Σ^{μ_ϵ}. The crucial point is that an analytic curve on Σ is mapped by f^{μ_ϵ} to an analytic curve on Σ^{μ_ϵ}. In general this is not true because the marking maps are quasiconformal.

Given two such trivializations we must show that the transition function is holomorphic. By the results of Hatcher and Thurston [24, 22], we know that any two pants decompositions are related by a sequence of two types of elementary moves. These two moves involve only genus-zero and genus-one surfaces. By using the associativity of the sewing isomorphism, the holomorphicity of the transition function can be reduced to the holomorphicity of the elementary moves. The genus-zero result is known (see Huang [30]).

So only the genus-one case remains. The methods used in [30] can be generalized to this case with the use of some classical results on Cauchy-type kernels and the Plemelj-Sokhotski formula for higher-genus Riemann surfaces which appear in [47, 52].

Because the trivializations were defined using pants decomposition the holomorphicity of the sewing operation is essentially built in. Combining the results up to this point results in Theorem 1.2.1.

We now turn to modular functors. Very briefly, a (holomorphic) *modular functor* is an assignment of a finite-dimensional complex vector space, $E(\Sigma)$, to each Riemann surface with *labeled* and parametrized boundaries. Morphisms are determined by the sewing operation. The vector spaces are required to form holomorphic vector bundle over the moduli space. We also assume that there are holomorphic projectively flat connections on these vector bundles. The determinant line bundle is the canonical example of a modular functor, where $E(\Sigma) = \mathrm{Det}_\Sigma$. Constructing modular functors is a highly non-trivial problem and is a major step in constructing a CFTs. The notion of a modular functor and its relation to CFT was given by Segal in [48].

The power of Schiffer variation should not be underestimated, for it produces a neighborhood in $\mathcal{M}(\Sigma)$ by performing only local variations on *genus-zero domains* of the Riemann surface. Variation of the local coordinates, or equivalently the boundary parametrization, only changes the surface in an annular neighborhood.

By using the sewing axiom for modular functors and the compatibility of the connection with sewing, the connection is shown to be determined by its restriction to the line bundle over the moduli space of disks and annuli. Line bundles over the moduli space of annuli are related to central extensions of $\mathrm{Diff}^+(S^1)$. An explicit construction of a connection is given in this case. Using the compatibility it is shown that the connections given by the modular functor in the disk and annulus cases are determined by such central extensions.

Chapter 2

Background on Complex Analysis, Teichmüller Theory and Schiffer Variation

This chapter collects some important definitions and results from the literature that will be used throughout the thesis. It is best just to refer back to these results as needed. Section 2.5 on marked holomorphic families of Riemann surfaces and the universality of the Teichmüller curve may contain unfamiliar results. Particular attention should also be paid to the Section 2.6 on Schiffer variation. In subsections 2.6.1 and 2.6.2 some results are formulated that do not appear explicitly elsewhere. They are such direct corollaries of known results that no originality is claimed by the author.

2.1 Notation and basic definitions

Let $\hat{\mathbb{C}}$ be the Riemann sphere which we also think of as the extended complex plane $\mathbb{C} \cup \{\infty\}$. Let Δ be the open unit disk centered at 0 in $\mathbb{C}$ (or $\hat{\mathbb{C}}$). Let Δ_∞ be the open unit disk centered at ∞ in $\hat{\mathbb{C}}$.

Let Σ be a Riemann surface. We say that Σ is of finite conformal type (g, n) if Σ is biholomorphically equivalent to $\hat{\Sigma} \backslash \{p_1, \ldots, p_n\}$, where $\hat{\Sigma}$ is a compact Riemann surface of genus g, and $p_1, \ldots, p_n$ are distinct points. Alternatively we can say that Σ can be compactified by the addition of n points. An ordering of the punctures is a bijection α from the set $\{1, \ldots, n\}$ to the set of punctures. Unless otherwise specified, Σ will always be of finite conformal type (g, n) with ordered punctures $p_1, \ldots, p_n$. When we want to make the punctures explicit we will write $(\Sigma, (p_1, \ldots, p_n))$.

Remark 2.1.1. Often it is easier think of a closed surface with marked points rather than punctures. For our purposes we really need the punctures to be included in the data of the surface as later we will represent these punctured surfaces as quotients of the upper half-plane by Fuchsian groups.

2.2 Facts from analysis and Riemann surface theory

2.2.1 Families of functions

The simple observation below in fact holds in much more generality as can be seen in Proposition 2.4.1. We record this just for later reference.

Proposition 2.2.1. *Let $f(t,x)$ be smooth as a function of (t,x) and analytic in t for each fixed x. Then the functions*

$$\frac{\partial^n f}{\partial x^n}(x,t),$$

for any n, are analytic in t for each fixed x.

Proof. As f is smooth the partial derivatives in x and $\bar{t}$ commute. □

The following basic fact has an easy direct proof using standard real analysis.

Proposition 2.2.2. *Let $f(t,x)$ be a smooth, complex-valued function on the domain $\Delta \times [0,1]$, where we recall Δ is the unit disk in $\mathbb{C}$. Assume $f(0,x) = 0$. Then $f(t,x)$ is uniformly bounded in x in the sense that $\forall \epsilon > 0, \exists \delta > 0$ such that $\max_{x\in[0,1]} |f(t,x)| < \epsilon, \forall |t| < \delta$.*

Remark 2.2.3. In the application of this result we think of t as a perturbation parameter. So the family $f(t,x)$ is a smooth perturbation of zero.

Proof. For $a > 0$, let

$$L_a = \{x \mid |f(t,x)| < \epsilon, \forall\, t \text{ such that } 0 < |t| < a\}$$

By joint continuity of f we know that the inverse image, $f^{-1}(B(0,\epsilon))$ is open. Thus L_a is open because it is the projection onto the x-axis of the intersection of $f^{-1}(B(0,\epsilon))$ and the strip $0 < |t| < a$.

Because $f(0,x) = 0$, we see that for any fixed x, $f(t,x) < \epsilon$ for t sufficiently small. Thus $\{L_a\}$ is an open cover of $[0,1]$. By compactness there exists a finite subcover $\{L_{a_i}\}, i = 1,\ldots,N$. Then for $|t| < \min a_i$ we have $|f(t,x)| < \epsilon$ for all $x \in [0,1]$ and this completes the proof. □

2.2.2 Hartogs' theorem in Banach space

The following is a deep theorem from several complex variables. A proof can be found in Hörmander ([25], Chapter 2.2).

Theorem 2.2.4 (Hartogs' Theorem). *If u is a complex valued function defined in the open set $\Omega \subset \mathbb{C}^n$ and u is analytic in each variable z_j when the other variables are given arbitrary fixed values, then u is analytic in Ω.*

For background on holomorphic functions in infinite-dimensions, see for example Mujica [43]. There are various generalizations of Hartogs' theorem to infinite-dimensions. The following version from Mujica [43], is sufficient for our purposes. A more general statement can be found in Kriegl and Michor [36].

Theorem 2.2.5. *Let $E_1, \ldots, E_n, F$ be Banach spaces, and let U be an open subset of $E_1 \times, \ldots, \times E_n$. Then a mapping $f : U \to F$ is holomorphic if and only if f is separately holomorphic.*

2.2.3 Holomorphic motions and the λ-lemma

The material is this section is taken from Astala and Martin [3] Originally the λ-lemma is due to Mañé, Sad and Sullivan [41].

Definition 2.2.1. Let A be a subset of $\hat{\mathbb{C}}$. A holomorphic motion of A is a map $f : \Delta \times A \to \hat{\mathbb{C}}$ such that

1. for any fixed $z \in A$, the map $t \to f(t, z)$ is holomorphic in Δ,

2. for any fixed $t \in \Delta$, the map $z \to f(t, z)$ is an injection and,

3. the mapping $f(0, z)$ is the identity on A.

Since t is a kind of deformation parameter we often use the notation $f_t(z)$ for $f(t, z)$. Also, as f_0 is the identity, we think of $f_t(z)$ as a holomorphic perturbation of the identity. The following theorem is the famous λ-lemma of Mañé, Sad and Sullivan [41]. It essentially says that any holomorphic perturbation of the identity must be a quasiconformal map.

Theorem 2.2.6. *If f is a holomorphic motion as above then f has an extension to $F : \Delta \times \overline{A} \to \hat{\mathbb{C}}$ such that*

1. *F is a holomorphic motion of $\overline{A}$*

2. *each $F_t(\cdot) : \overline{A} \to \hat{\mathbb{C}}$ is quasiconformal*

3. *F is jointly continuous in (t, z)*

2.2.4 Automorphisms

Let $\mathrm{Aut}(\Sigma)$ be the space of biholomorphisms from the Riemann surface Σ to itself. Let $\mathrm{Aut}_0(\Sigma)$ be the subspace $\mathrm{Aut}(\Sigma)$ consisting of biholomorphisms which are homotopic to the identity.

Theorem 2.2.7. *If Σ is of finite conformal type (g, n) and $2g-2+n > 2$, then $\mathrm{Aut}_0(\Sigma)$ contains only the identity. In other words there are no non-trivial automorphisms that are homotopic to the identity.*

Outline of Proof. A stronger theorem is proved in Farkas and Kra [12, Section V.3] which essentially says that for $T \in \mathrm{Aut}(\Sigma)$, if T_* acts as the identity on the first homology H_1 then $T = Id$. By homotopy invariance we know that if T is homotopic to the identity then $T_* = Id$.

A short proof using some facts from hyperbolic geometry is given in McMullen [42, Theorem 2.2]. □

Theorem 2.2.8. *If Σ is of finite conformal type (g, n) and $2g - 2 + n > 0$ then $\mathrm{Aut}(\Sigma)$ is a finite group.*

See Farkas and Kra [12, Section V.1] for a proof. Their result is for surfaces without punctures and thus $g \geq 2$. The extra cases for genus-zero surfaces with $n > 2$ and genus-one surfaces with $n > 0$ can be easily treated directly.

2.3 Teichmüller space

This section closely follows the presentation in Nag [44].

The theory of quasiconformal mappings is fundamental to the development of the analytic theory of Teichmüller space. Consequently quasiconformal mappings play an important role throughout this work. The reader is referred to Appendix A and references therein for the appropriate background material.

A key result is that the Teichmüller space, $T(\Sigma)$, is a finite dimensional complex manifold if and only if Σ is of finite conformal type. Specifically, if Σ is of conformal type (g, n), then $T(\Sigma)$ has complex dimension $3g - 3 + n$ provided that $2g - 2 + n > 0$. This last condition only excludes $g = 0, n = 0, 1, 2$ and $g = 1, n = 0$ which can easily be treated directly. Note that the excluded cases are exactly those for which the automorphism group of the surface is continuous.

From now on we only consider the case of surfaces of finite conformal type (g, n) although most of the definition immediately generalize.

Definition 2.3.1. A *marked Riemann surface* modelled on Σ is a triple (Σ, f, Σ_1), where Σ_1 is a Riemann surface and $f : \Sigma \to \Sigma_1$ is a quasiconformal homeomorphism called the marking map.

We now give the punctures on the base space Σ a fixed ordering α, and we label the punctures $(x_1, \ldots, x_n)$. We set the order of the punctures on Σ_1 to be that induced from α by f.

Remark 2.3.1. We would like to write $p_i = f(x_i)$ but to do this we really need to consider a compactification of Σ and the extension of f. There is no problem doing this but it is unnecessary at this point.

Definition 2.3.2. Let $\hat{M}(\Sigma)$ be the collection of marked Riemann surfaces (Σ, f, Σ_1).

Definition 2.3.3. Two marked Riemann surfaces (Σ, f, Σ_1) and (Σ, g, Σ_2) in $\hat{M}$ are called *Teichmüller equivalent* ($\sim$) if and only if there exists a biholomorphism $\sigma : \Sigma_1 \to \Sigma_2$ such that $g^{-1} \circ \sigma \circ f : \Sigma \to \Sigma$ is homotopic to the identity.

The following diagram may help to clarify this definition.

$$\begin{array}{ccc} \Sigma & \xrightarrow{\simeq \mathrm{id}} & \Sigma \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ \Sigma_1 & \xrightarrow{\sigma} & \Sigma_2 \end{array} \tag{2.1}$$

Note that the homotopy condition implies that $g^{-1} \circ \sigma \circ f$ preserves the order of the punctures on Σ, and consequently σ is order preserving.

From Theorem 2.2.7 it follows that if $2g - 2 + n > 0$ then whenever such a σ exists it is unique.

Definition 2.3.4. The *Teichmüller space*, $T(\Sigma)$, of Σ is $\hat{M}(\Sigma)/\sim$. Equivalence classes are written $[\cdot, \cdot, \cdot]$.

Although we do not use this explicitly it is worth noting the important fact that $T(\Sigma)$ is a complete metric space with respect to the *Teichmüller metric*

$$\tau([\Sigma, f, \Sigma_1], [\Sigma, g, \Sigma_2]) = \inf_{\sigma}\{\frac{1}{2}\log K(\sigma)\}$$

where the infimum is taken over all quasiconformal homeomorphisms $\sigma : \Sigma_1 \to \Sigma_2$ that are homotopic to $g \circ f^{-1}$. See Appendix A for the definition of the dilation, K, of a quasiconformal map.

Theorem 2.3.2. *If Σ is of conformal type (g, n) where $2g - 2 + n > 0$, then $T(\Sigma)$ is a complex manifold of complex dimension $3g - 3 + n$*

There is an intermediate space which is of interest in understanding the complex structure of Teichmüller space. Let $M(\Sigma) = \hat{M}/\approx$ where $(\Sigma, f, \Sigma_1) \approx (\Sigma, g, \Sigma_2)$ if and only if $g \circ f^{-1}$ is conformal.

The projection

$$\Phi : M(\Sigma) \longrightarrow T(\Sigma)$$

is called the *fundamental projection.* There is a sequence of natural projections

$$\hat{M} \longrightarrow M(\Sigma) \longrightarrow T(\Sigma).$$

Recall from Appendix A that $L^\infty_{(-1,1)}(\Sigma)_1$ is the unit ball in the space of Beltrami differentials. Assume that $\Sigma = U/G$. Given $\mu \in L^\infty_{(-1,1)}(\Sigma)_1$ there exists a unique normalized quasiconformal map w^μ. Using this map to deform the Fuchsian group we obtain a new Riemann surface $w^\mu(U)/G^\mu$. See Section A.3 for details.

Lemma 2.3.3. *There is a canonical identification of $M(\Sigma)$ with $L^\infty_{(-1,1)}(\Sigma)_1$.*

Outline of Proof. See Nag [44, page 106] for a complete proof. The identification is achieved in the following manner. Given any $(\Sigma, f, \Sigma_1) \in \hat{M}$, the complex dilation of f is a Beltrami differential. That is, $\mu_f \in L^\infty_{(-1,1)}(\Sigma)_1$.

Conversely, given any $\mu \in L^\infty_{(-1,1)}(\Sigma)_1$ we have $(\Sigma, f^\mu, \Sigma^\mu) \in \hat{M}$. □

Theorem 2.3.4. *The fundamental projection*

$$\Phi : L^\infty_{(-1,1)}(\Sigma)_1 \longrightarrow T(\Sigma)$$

is holomorphic.

By using this very important result, holomorphicity questions in Teichmüller space can sometimes be reduced to questions about holomorphic families of differentials in $L^\infty_{(-1,1)}(\Sigma)_1$.

2.3.1 Change of base point

The following discussion follows Nag [44, pages 122-3 and 186].

Teichmüller space is a "moduli space" of conformal equivalence classes of surfaces of a fixed topological type. The role of Σ should just be to specify the topological type (g, n), and so the Teichmüller space $T(\Sigma)$ should not depend on the choice of reference surface. This can be made precise. We call $[\Sigma, id, \Sigma]$ the base point of $T(\Sigma)$.

Let $h : \Sigma' \to \Sigma$ be a quasiconformal homeomorphism. Such a map induces a map

$$h^* : T(\Sigma) \to T(\Sigma')$$

given by $h^*([\Sigma, f, \Sigma_1]) = [\Sigma', f \circ h, \Sigma_1]$. This has produced a *change of base point* for Teichmüller space. It is not hard to prove that h^* is an isometric isomorphism from $T(\Sigma)$ to $T(\Sigma')$. As we are interested in the complex structure on Teichmüller space we need to know what happens to the complex structure under h^*.

Theorem 2.3.5. *h^* is a biholomorphism*

Proof. See Nag [44], page 186. □

Remark 2.3.6. In Nag this fact is used to prove that the complex structure on $T(\Sigma)$ is canonical in the sense that it does not depend on the base point. So we can write $T(g,n)$ for the Teichmüller space of Riemann surfaces of finite conformal type (g,n).

It is worth noting that such an h can be used to change the order of the punctures on the base surface. This follows from the existence of quasiconformal maps $\Sigma \to \Sigma$ permuting the punctures in all possible ways.

2.3.2 Moduli space and the mapping class group

The (Riemann) moduli space is the space of conformal equivalence classes of Riemann surfaces. This space is the one we are ultimately interested in. Since we are considering surfaces with ordered punctures, the conformal equivalence must preserve the order.

Let Σ be a Riemann surface of finite conformal type (g,n) and α an ordering of the punctures.

Definition 2.3.5. The *moduli space* $\mathcal{M}(\Sigma)$ of Riemann surfaces with ordered punctures, modelled on Σ, is defined to be $\hat{M}(\Sigma)/\sim_R$, where $(\Sigma, f, \Sigma_1) \sim_R (\Sigma, g, \Sigma_2)$ if and only if there exists a biholomorphism $\sigma : \Sigma_1 \to \Sigma_2$ that preserves the order of the punctures.

It is not hard to see that this indirect definition of the moduli space is equivalent to the usual one.

In Teichmüller theory the mapping class group is often referred to as the (Teichmüller) modular group and is denoted $\mathrm{Mod}(\Sigma)$. There are several definitions which are equivalent when the surface is compact. Also there are various "mapping class groups" that differ in the sense they preserve the boundary and punctures.

We use a different mapping class group to Nag [44], in that we require the order of the punctures to be preserved. See for example Harer [21]. This is often called the *pure mapping class group*. Using this group means that the moduli space we obtain will be that of order-preserving conformal equivalence classes.

Let $PQ(\Sigma)$ be the group of all orientation preserving quasiconformal homeomorphisms of Σ to itself that preserve the order of the punctures. Let $PQ_0(\Sigma)$ be the subgroup of $PQ(\Sigma)$ consisting of maps which are homotopic to the identity.

Definition 2.3.6. The *pure mapping class group* is the quotient

$$\mathrm{PMod}(\Sigma) = PQ(\Sigma)/PQ_0(\Sigma).$$

One of the important results on the mapping class group is that it is finitely generated by Dehn twists. This fact will be useful in defining the action of the mapping class group on the determinant line bundle. See Section 5.8.

From Section 2.3.1 we see that the mapping class group acts on Teichmüller space by

$$\rho \cdot [\Sigma, f, \Sigma_1]) = [\Sigma, f \circ \rho, \Sigma_1]$$

where $\rho \in PQ(\Sigma)$.

At least for our purposes, the fundamental use of the mapping class group is that taking the quotient of Teichmüller space by this group produces the moduli space. We state this formally in the next theorem, a proof of which can be found in Nag [44, sections 2.7 and 3.2.5].

Theorem 2.3.7. *If Σ is of finite conformal type then $\mathrm{PMod}(\Sigma)$ acts properly discontinuously as biholomorphic automorphisms on $T(\Sigma)$. Moreover the quotient can be naturally identified with the moduli space $\mathcal{M}(\Sigma)$.*

Actually the theorem in Nag [44] is stated for the usual, that is not pure, mapping class group. The result in our case immediately holds as the pure mapping class group is a subgroup of the usual mapping class group.

2.4 Technical results on quasiconformal maps

The following collection of results has not been put in Appendix A as it is not standard material in the theory of quasiconformal maps. The reader should consult Appendix A for notation and definitions used in this section.

This material is taken from the proof of Theorem 2.5.1 given in Nag [44], Section 5.1. The results go back at least to Bers [8].

Proposition 2.4.1. (Nag [44, pages 315-7]). *Let $W(t,x)$ be a complex valued function for $|t| < R$ and $x \in (a,b) \subset \mathbb{R}$. Suppose that $W(t,x)$ is holomorphic in t for each fixed x, and is C^1 in x for each fixed t. Suppose further that $|\frac{\partial W}{\partial x}| \leq B$ for all t, and for x in a compact set K of the $x-$domain (B may depend on K). Then $\frac{\partial W}{\partial x}$ is also holomorphic in t for each fixed x. Moreover, W and $\frac{\partial W}{\partial x}$ are jointly continuous in t and x*

Proposition 2.4.2. (Nag [44, page 317]). *Let $t \mapsto \sigma_t$ be a complex one-parameter family of Beltrami differentials. That is, $\sigma_t \in L^\infty(U,G)_1$ and σ_t depends holomorphically on t in $|t| < R$. Then one can construct another one-parameter family $t \mapsto \mu_t$, again holomorphically dependent on t in $|t| < R_0$ (some $0 < R_0 \leq R$), so that (i) $\mu_t(z)$ is real analytic in z on U, (ii) μ_t and it's partial derivatives with respect to z and $\bar{z}$ up to order n are bounded, (independent of t), on compact z sets, and (iii) $[\mu_t] = [\sigma_t]$ in $T(G)$.*

Proposition 2.4.3. (Nag [44, pages 35 and 37]) *Any quasiconformal map whose complex dilation is real analytic must itself be real analytic. An analogous statement for class C^k also holds.*

Proposition 2.4.4. (Nag [44, pages 318 and 319]). *Assume $\mu_t(z)$ is real analytic (in z), holomorphic in t (for $|t| < R$) and $||\mu_t|| \leq k < 1$, for $|t| < R$. Let $W(t,z) = w^{\mu_t}(z)$ be the normalized solution to the Beltrami equation. Then the partial derivatives $\frac{\partial W}{\partial z}$ and $\frac{\partial W}{\partial \bar{z}}$ are uniformly bounded on compact z sets by a bound depending on k and the compact set, but not on t.*

Remark 2.4.5. By Proposition 2.4.1 we immediately get that $\frac{\partial W}{\partial z}$ and $\frac{\partial W}{\partial \bar{z}}$ depend analytically on t.

The next proposition has a simple but tricky proof. The result itself is surprising and useful in our later work. The notation has been changed from Nag [44] in order to apply the result in a different context.

Proposition 2.4.6. *Let $W(t,z)$ and $V(t,z)$ be complex valued functions on the domain, $|t| < R$ and $z \in K$, (K compact), satisfying the following conditions.*

Basic conditions:

1. *Both V and W are homeomorphisms in z for each fixed t.*

2. *Both V and W are analytic in t for each fixed z.*

3. *$G(t,z) = (V \circ W^{-1})(t,z)$ is analytic (and therefore biholomorphic) in z for each fixed t, where W^{-1} is the inverse in the z-variable.*

Technical conditions:

1. *W is real analytic in z for each fixed t.*

2. *$\frac{\partial W(t,z)}{\partial z}$ and $\frac{\partial W(t,z)}{\partial \bar{z}}$ are uniformly bounded on compact z sets, independently of t.*

3. *$V(t,z)$ is uniformly bounded on compact z sets, independently of t.*

Then $G(t,z) = (V \circ W^{-1})(t,z)$ is analytic as a function of t for each fixed z.

Remark 2.4.7. Note that it is not in general true that $W^{-1}(t,z)$ is analytic in t as the example $W(t,z) = z + t\bar{z}$ shows.

Proof. Cauchy's theorem will be used to express $G(t,z)$ without using V^{-1}. Choose arbitrary $w_0 \in K$ and t_0 such that $|t_0| < R$. Let $z_0 = W(t_0, w_0)$ and pick $r > 0$ such that the ball $B(w_0, r)$ is contained in K.

Let $\gamma_t(\theta) = W(t, w_0 + re^{i\theta})$ be the curve that is the image of the circle $B(w_0, r)$ under W. For t sufficiently close to t_0 and by the conditions on W we see that γ_t is a closed rectifiable curve around z_0.

Since $G(t,z)$ is biholomorphic in z, Cauchy's theorem gives

$$\begin{aligned} G(t,z_0) &= \frac{1}{2\pi i}\int_{\gamma_\theta} \frac{G(t,z)}{z-z_0}dz \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{V(t,re^{i\theta})}{W(t,e^{i\theta})-z_0}\frac{\partial W(t,re^{i\theta})}{\partial \theta}d\theta. \end{aligned} \tag{2.2}$$

Let $I(t,z)$ be the integrand in Equation 2.2. The conditions for Proposition 2.4.1 are satisfied by $W(t,z)$ and thus its partial derivatives are analytic in t (the partial in θ can be expressed in term of partials in z and $\bar{z}$). Therefore $I(t,z)$ is analytic in t.

To conclude $G(t, z_0)$ is analytic in t we need to move derivatives in t past the integral. By a standard application of the Lebesgue dominated convergence theorem, it is sufficient to show that $\frac{\partial I(t,z)}{\partial t}$ is uniformly bounded.

By the technical assumptions in the hypothesis of the theorem we know $I(z,t)$ is uniformly bounded. Now recall that for any analytic function $f(z)$,

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(\xi)}{(\xi - z)^2} d\xi,$$

and so a uniform bound on the $f(z)$ gives a uniform bound on $f'(z)$. Applying this to $I(t,z)$, as a function of t, gives the desired bound on $\frac{\partial I(t,z)}{\partial t}$.

Thus, for t sufficiently close to t_0, $G(t, z_0)$ is analytic as a function of t . Since z_0 and t_0 are arbitrary $G(t,z)$ is analytic in t, for $|t| < R$ and for each $z \in K$. □

Remark 2.4.8. If $f(z)$ is any analytic function of z such that $G(t,z) = (W \circ f \circ V^{-1})(t,z)$ makes sense then $G(t,z)$ will be analytic in t. The proof above needs only to be trivially modified.

Remark 2.4.9. If $G(t,z)$ is also injective as a function of t then it will be biholomorphic, as a function of two complex variables, by Hartogs' theorem (see Theorem 2.2.4).

2.5 Universal families over Teichmüller space

This material is from chapter five of Nag [44]. See Appendix A for the basics of quasiconformal mappings and the Teichmüller group $T(G)$. We will construct a fiber space over $T(G)$ where the fiber over $[\mu]$ will be a Riemann surface that is represented by the point $[\mu]$ in Teichmüller space. The representative surfaces will be chosen so that the family forms a holomorphic fiber space. One way to think about this is as an analogue of the classical genus-zero uniformization theorem for Riemann surfaces. An advantage of this approach is that rather than working with equivalence classes we can instead work with the canonical representatives.

2.5.1 The Bers fiber space

Definition 2.5.1. Let G be an arbitrary Fuchsian group. Define the *Bers fiber space*

$$F : F(G) \longrightarrow T(G)$$

by

$$F([\mu], z) = [\mu],$$

where $F(G) = \{([\mu], z) \in T(G) \times \mathbb{C} \,|\, \mu \in L^\infty(U, G)_1, z \in w^\mu(U)\}$

The fiber above $[\mu]$ is $w^\mu(U)$, which we know by Appendix A, Lemma A.3.7, only depends on the equivalence class $[\mu]$ of μ. As a subset of $T(G) \times \mathbb{C}$, $F(G)$ becomes a complex Banach manifold. Note that topologically $F(G)$ is homeomorphic to $T(G) \times U$ and so is a trivial bundle. However this is not true complex analytically.

The following theorem contains the heart of the matter. The holomorphicity result has a non-trivial proof which relies on some estimates from elliptic partial differential equation theory and application of the technical results outlined in Section 2.4. This is originally due to Bers, [8]. See also Nag, [44]. See Appendix A for definitions and notation.

Theorem 2.5.1. *The extended modular group mod(G) anti-acts, as a group of biholomorphic automorphisms on $F(G)$, inducing the Mod(G) action on the base space $T(G)$. The subgroup G in mod(G) acts as fiber-preserving automorphisms of $F(G)$. The action of $\omega \in N_{q.c.}(G)$ is given by*

$$\omega^*([\mu], z) = ([\nu, z_1])$$

where $[\nu] = \omega^*([\mu])$, and

$$z_1 = w^\nu \circ \omega^{-1} \circ (w^\mu)^{-1}(z), \quad z \in U^\mu, \quad z_1 \in U^\nu.$$

Here ν is the complex dilation of $w_\mu \circ \omega$.

Showing that the map $z \mapsto z_1$ is analytic is the technical part of the proof.

2.5.2 The fiber space $V(G)$

First we will give a general definition of a holomorphic fiber space.

Definition 2.5.2. A *holomorphic fiber space* is a pair (V, B) of complex manifolds and a map $\pi : V \to B$, such that π is a holomorphic, submersive, and surjective map.

Here a submersive map is considered to be 'split' in the sense that the kernel of the derivative of π is a direct summand everywhere. See Nag [44], Section 1.6.5, for additional background. A consequence of this is that π always possesses holomorphic local sections.

To construct the fiber space $V(G)$ we will take the quotient of $F(G)$ by G. The fiber over the point $[\mu]$ will be $w^{\mu}(U)/G^{\mu}$.

Theorem 2.5.2. *Let G be an arbitrary Fuchsian group. As a subgroup of mod(G), G acts properly discontinuously on the holomorphic fiber space $F(G)$ as a group of biholomorphic fiber-preserving automorphisms. Therefore, if G is torsion free, the quotient $F(G)/G = V(G)$ is a holomorphic fiber space over $T(G)$, with fiber above $[\mu]$ being $\Sigma^{\mu} = w^{\mu}(U)/G^{\mu}$. In fact $F(G)$ is the universal cover of $V(G)$.*

The proof is not difficult as the non-trivial work has been done in Theorem 2.5.1. Details can be found in Bers [8], or Nag [44].

We now specialize to the case of interest where the Riemann surface, Σ, is of finite conformal type (g, n) with $2g - 2 + n > 0$. Then $\Sigma = U/G$ for some finitely generated, torsion-free Fuchsian group, G, of the first kind (see Proposition A.3.1). Recall that $T(G)$ is finite dimensional in this case and is often denoted $T(g, n)$.

Definition 2.5.3. The *n-punctured universal family* over $T(g, n)$ is defined to be the holomorphic fiber space

$$V(G) \xrightarrow{\pi} T(g, n).$$

Remark 2.5.3. Topologically this bundle is the globally trivial bundle $T(G) \times \Sigma$. Holomorphically the bundle is not even locally trivial because for each different $[\mu]$ the surfaces Σ^{μ} are not biholomorphically equivalent. So we do not have a fiber bundle but only a fiber space.

The Riemann moduli space $R(G)$ can also be considered in this context.

Proposition 2.5.4. *If $T(G)$ is finite dimensional then $mod(G)$ acts properly discontinuously on $F(G)$.*

Corollary 2.5.5. *If G is any finitely generated Fuchsian group of the first kind, then*

$$F(G)/mod(G) \equiv V(G)/Mod(G) \xrightarrow{p} T(G)/Mod(G) \equiv R(G)$$

with p a holomorphic surjection.

2.5.3 Marked families of Riemann surfaces

As well as the universal family $V(G)$ we will need to consider other holomorphic families of Riemann surfaces. The following material is taken from Nag [44, Chapter 5.4].

Let $p : E \to B$ be a holomorphic fiber space.

Definition 2.5.4. A *holomorphic family of complex manifolds* is a holomorphic fiber space such that p is topologically a locally-trivial fiber bundle with fiber model some topological space X.

If each fiber $p^{-1}(b)$ is a Riemann surface then we say that $p : E \to B$ is a *holomorphic family of Riemann surfaces.*

Consider now a holomorphic family of Riemann surfaces where the fiber model is a Riemann surface, Σ, of topological type (g, n). Each fiber is diffeomorphic to Σ. General theory says that the fiber space is actually a locally-trivial C^∞ fiber space.

The trivializations are expressed in the diagram

$$\begin{array}{ccc} p^{-1}(U_\alpha) & \xrightarrow{F_\alpha} & U_\alpha \times \Sigma \\ & {\scriptstyle p}\searrow \quad \swarrow & \\ & U_\alpha \subseteq B & \end{array}$$

where the sets U_α cover B and the F_α are C^∞ diffeomorphisms.

Let $U_{\alpha\beta} = U_\alpha \cap U_\alpha$. For $v \in U_{\alpha\beta}$ we have

$$\begin{cases} F_\alpha(v) = (p(v), g_\alpha(v)) \\ F_\beta(v) = (p(v), g_\beta(v)) \end{cases}$$

The trivializations can be chosen such that the g_α are orientation preserving. For each $b \in B$ we have the transition diffeomorphism

$$g_{\alpha\beta}(b) = g_\alpha \circ g_\beta^{-1} : \Sigma \longrightarrow \Sigma.$$

In the language of vector bundles we say that the structure group is $\mathrm{Diff}^+(\Sigma, n)$.

Definition 2.5.5. A *marking* of the holomorphic family of Riemann surfaces is a reduction of the structure group to $\mathrm{Diff}_0^+(\Sigma, n)$. We call such a fiber space a *marked n-punctured family.*

In more concrete terms note that for any $b \in B$ we have

$$[\Sigma, g_\alpha^{-1}|_b, p^{-1}(\Sigma)] \in T(\Sigma).$$

Thus to the marked n-pointed family (E, p, B) we have a canonical *classifying* map

$$h : B \to T(\Sigma).$$

Definition 2.5.6. A morphism of marked n-pointed families is a pair of biholomorphic maps (ζ, η)

$$\begin{array}{ccc} E_1 & \xrightarrow{\zeta} & E_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B_1 & \xrightarrow[\eta]{} & B_2 \end{array}$$

such that the diagram commutes, ζ restricted to each fiber is a biholomorphism of Riemann surfaces, and (ζ, η) is topologically a $\mathrm{Diff}_0^+(\Sigma, n)$ bundle map.

We can now state the universality of $V(G)$. This theorem is of considerable use in our later work.

Theorem 2.5.6. *Let $2g-2+n>0$. Given any marked n-punctured holomorphic family (E,p,B), of genus g Riemann surfaces, there is a unique morphism of marked families*

$$\begin{array}{ccc} E & \xrightarrow{H} & V(g,n) \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle \pi} \\ B & \xrightarrow[h]{} & T(g,n) \end{array}$$

where h is the classifying map defined above.

2.6 Schiffer variation

As discussed in the introduction, Schiffer variation is one of the key concepts in proving results about the sewing operation and the determinant line bundle. Moreover, the proof of the holomorphicity of the sewing operation is inspired by Schiffer variation. The power of this method is that a holomorphic coordinate chart can be produced in Teichmüller space by performing only local operations on the Riemann surface. Therefore all local deformations of complex structure can be achieved without affecting the punctures and local coordinates.

The basic idea of Schiffer variation is simple. A small "disk" is removed from the Riemann surface and the hole is filled by sewing in a disk which has been deformed. The deformation is determined by a complex parameter ϵ. These parameters turn out to provide local coordinates for Teichmüller space.

We want to formulate the sewing operation in a standard way that is compatible with the sewing of Riemann surfaces with parametrized boundary. So we use different notation to Nag [44], as not to completely confuse the issue.

Let ζ be local holomorphic co-ordinate around a point $p \in \Sigma$. Assume that $\zeta(p) = 0$ and $\overline{B(0,r)} \subset \operatorname{Im}\zeta$ for some $r > 1$. Let $B = B(0,1)$ and let $D = \zeta^{-1}(B)$

We now define the important function which essentially determines the variation. Let $V \subset \operatorname{Domain}(\zeta)$ be an annular neighborhood of ∂D, and let $A = \zeta(V)$. Note that

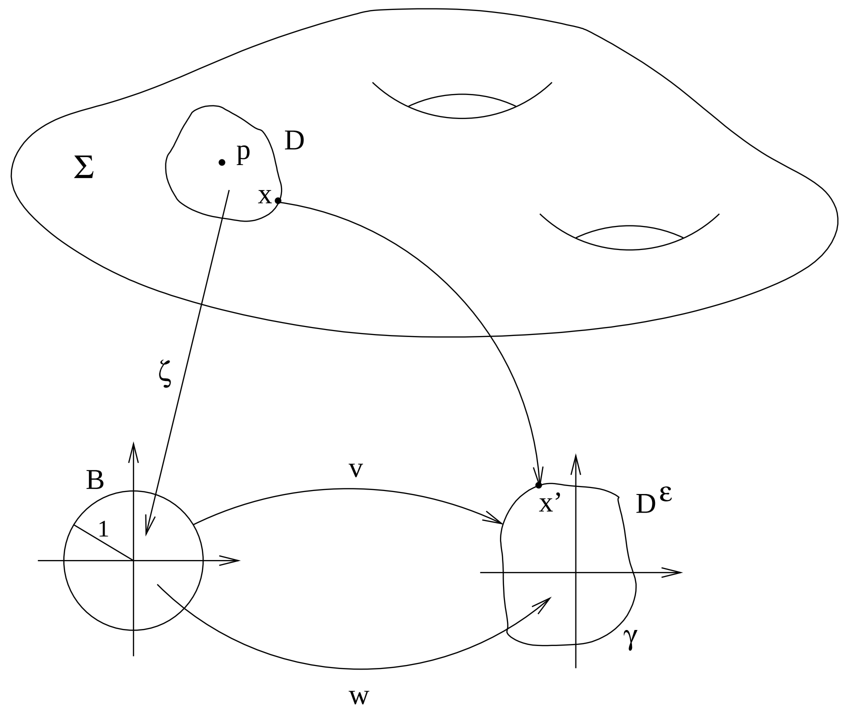


Figure 2.1: Schiffer variation

A is an annular neighborhood of the unit circle. Let

$$v(z) = z + \frac{\epsilon}{z} : A \longrightarrow \mathbb{C}$$

For ϵ sufficiently small v is biholomorphic. We can also choose ϵ such that $v(\partial B) \in A$ and therefore some neighborhood $A_1 \subset A$ of ∂B is mapped into A.

We now define the manifold (open set $\in \mathbb{C}$) that we will sew into $\Sigma \setminus D$. Let $\gamma = v(\partial B)$ and let D^ϵ be the interior of the Jordan curve γ. We regard D^ϵ as a manifold (with the standard complex structure induced by $\mathbb{C}$) with analytic boundary parametrization given by v^{-1}. See Figure 2.1. We now identify the boundaries of $\Sigma \setminus D$ and D^ϵ by identifying $x \in \partial D$ with $x' \in \partial D^\epsilon$ if and only if $x' = (v \circ \zeta)(x)$. Let

$$\Sigma^\epsilon = (\Sigma \setminus D) \sqcup D^\epsilon \ / \text{ boundary identification}$$

and we say this surface is obtained from Σ by *Schiffer variation* of complex structure on D. To see that this is well defined we must show that if x and x' are identified then $\zeta(x) = v^{-1}(x')$. This is easy as $v^{-1}(x') = v^{-1}((v \circ \zeta)(x)) = \zeta(x)$. Drawing a picture makes all this clear. Basically a small 'half-ball' with center x' in D^ϵ and a similar 'half-ball' with center x in $\Sigma \setminus D$ are mapped to a whole ball, with center $\zeta(x) \in B$, in A under the boundary parametrizations ζ and v^{-1}. A detailed description of the local charts on Σ^ϵ is given in Section 2.6.1.

Remark 2.6.1. Here we have followed the construction given in Nag. In Gardiner [20] it looks a little different in that instead of considering $D^\epsilon \subset \mathbb{C}$, a subset of the original manifold is used. In our case this corresponds to $\zeta^{-1}(D^\epsilon)$.

Remark 2.6.2. In the above construction the orientation reversing of the boundary identification is hidden in the fact that D^ϵ maps to the interior of B whereas a neighborhood of the boundary in $\Sigma \setminus D$ maps to the exterior. So the map between local coordinate neighborhoods is given by $v \circ Id \circ \zeta$. In other words, the map in the complex plane is just the identity. It is easy to change the definition of the local co-ordinate on D^ϵ so that we have the inversion map $\mathbb{J}(z) = 1/z$ in the complex plane as in Huang [30]. Doing this would also fit the sewing described in Ahlfors and Sario [2].

We now construct a quasiconformal homeomorphism $\nu^\epsilon : \Sigma \to \Sigma^\epsilon$. This will be done piecewise. On $\Sigma \setminus D$, it is just the identity. Let

$$w(z) = z + \epsilon\bar{z} : \overline{B} \longrightarrow \overline{D^\epsilon}.$$

On the boundary $v = w$ as $z = \exp(i\theta)$. It is easy to see that w is injective (and linear). So w is a homeomorphism from $\overline{B}$ to $\overline{D^\epsilon}$. The full definition of ν^ϵ is:

$$\nu^\epsilon(x) = \begin{cases} x & x \in \Sigma \setminus D \\ (w \circ \zeta)(x) & x \in D \end{cases} \tag{2.3}$$

To see that this is map is quasiconformal we must compute its complex dilation $\mu(\zeta^\epsilon) \in L^\infty_{(-1,1)}(\Sigma)_1$. In terms of local co-ordinates, $\nu^\epsilon = Id \circ (w \circ \zeta) \circ \zeta^{-1} = w$. Direct computations gives

$$\mu(w) = \frac{\partial_{\bar{z}} w}{\partial_z w} = \epsilon$$

So

$$\mu(\nu^\epsilon) = \begin{cases} 0 & \text{on } \Sigma \setminus D \\ \epsilon \frac{d\bar{\zeta}}{d\zeta} & \text{on } D \end{cases} \tag{2.4}$$

from which we see that $||\mu(\nu^\epsilon)||_\infty = |\epsilon|$. Thus ν^ϵ is quasiconformal for $|\epsilon| < 1$, and moreover we see that $\mu(\nu^\epsilon)$ depends holomorphically on ϵ. So we now have a point $[\Sigma, \nu^\epsilon, \Sigma^\epsilon] \in T(\Sigma)$ obtained by Schiffer variation of the base point $[\Sigma, id, \Sigma]$.

We now give notation for Schiffer variation on n disjoint disks. Let $(D_1, \ldots, D_n)$ be n disjoint parametric disks on Σ, where $D_i = (\zeta_i)^{-1}(B(0,1))$ for suitably chosen local coordinates ζ_i. Let $D = D_1 \cup \cdots \cup D_n$ and let $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \mathbb{C}^n$. Schiffer variation can be performed on the n disks to get a new surface which we again denote by Σ^ϵ. The map ν^ϵ becomes

$$\nu^\epsilon(x) = \begin{cases} x & x \in \Sigma \setminus D \\ (w \circ \zeta_i)(x) & x \in D_i \ , i = 1, \ldots, n \end{cases} \tag{2.5}$$

The following theorem is one of the main result on Schiffer variation. The current formulation is due to Gardiner [20] and Nag [44]. Some history and references are given in both.

Theorem 2.6.3. *For Schiffer variation on n disjoint disks, the map S,*

$$(\epsilon_1, \ldots, \epsilon_n) \overset{S}{\longmapsto} \Sigma^\epsilon$$

is holomorphic from a neighborhood of zero in $\mathbb{C}^n$ into $T(\Sigma)$. Let $T(\Sigma)$ have finite complex dimension, d.(i.e. assume $2g - 2 + n > 0$ and let $d = 3g - 3 + n$). Then the following assertions are valid:

1. *Given any d points $\{p_1 \ldots p_d\}$ on Σ it is possible to choose parametric unit disks with centers $\{p'_1 \ldots p'_d\}$ lying in arbitrarily small neighborhoods of the original points so that the variation parameters $(\epsilon_1, \ldots, \epsilon_d)$ are holomorphic co-ordinates for $T(\Sigma)$ around Σ.*

2. *If we specify d disjoint parametric unit disks on Σ with boundaries $\{\beta_1, \ldots, \beta_d\}$, it is possible to choose local parameters for these disks so that the corresponding ϵ's again provide holomorphic local coordinates on $T(\Sigma)$.*

For our purposes it is important to note that the disks where the variation is performed can be performed essentially arbitrarily.

This construction produces a neighborhood of the identity $[\Sigma, id, \Sigma] \in T(\Sigma)$. Since any point can be considered a base point for Teichmüller space this procedure will give local coordinates at all points. See Section 2.6.2 for details.

2.6.1 Local coordinates description

At times it will be more useful for our purposes to consider Schiffer variation as changing the complex structure on the original surface. This way, all the varied surfaces are topologically identical. Although this is not explicitly written down in Nag [44] or Gardiner [20], it is a straight forward exercise.

The quasiconformal homeomorphism $\nu^\epsilon : \Sigma \to \Sigma^\epsilon$ can be used to pull back the complex structure from Σ^ϵ to Σ. The topological surface Σ with this complex structure will be denoted $\Sigma_{\mu(\nu^\epsilon)}$. This notation comes from the following way of describing this Riemann surface. The complex dilation $\mu(\nu^\epsilon) \in L^\infty_{(-1,1)}(\Sigma)$ produces a complex structure on Σ in a canonical way. See Section A.3.1 for the general construction. It is not hard to show that these two procedures describe identical complex structures on Σ.

An explicit description of the local coordinates on $\Sigma_{\mu(\nu^\epsilon)}$ will now be given by just pulling back the charts on Σ^ϵ using ν^ϵ.

Consulting Figure 2.2 may help one to understand the notation that follows. Let $0 < r_0 < 1$, and let $r_1 > 1$ be such that $B(0, r_1) \subset \operatorname{Im} \zeta$. This is possible by our initial assumptions. Let $D_1 = \zeta^{-1}(B(0, r_1))$, and let $D_0 = \zeta^{-1}(B(0, r_0))$. Let the charts on $\Sigma \backslash \overline{D}$ be the same as the original charts on Σ, say $\{(U_\alpha, \zeta_\alpha)\}$. In particular, note that on $D_1 \backslash \overline{D}$ the coordinate is ζ. On D the local coordinate is $w^\epsilon \circ \zeta$.

At this point only the boundary of D is not covered by charts. Let $\{W_i\}, i = 1, \ldots, n$ be open domains contained in the annulus $B(0, r_1) \backslash B(0, r_0)$ that cover the boundary of $B(0, 1)$ (actually $n = 2$ is enough). Let $V_i = \zeta^{-1}(W_i)$. Clearly V_i cover ∂D. Let $V_i^+ = V_i \cap (D_1 \backslash D)$ and let $V_i^- = V_i \cap \overline{D}$. Note that V_i^+ is just that part of V_i outside D, and V_i^- is just the part inside.

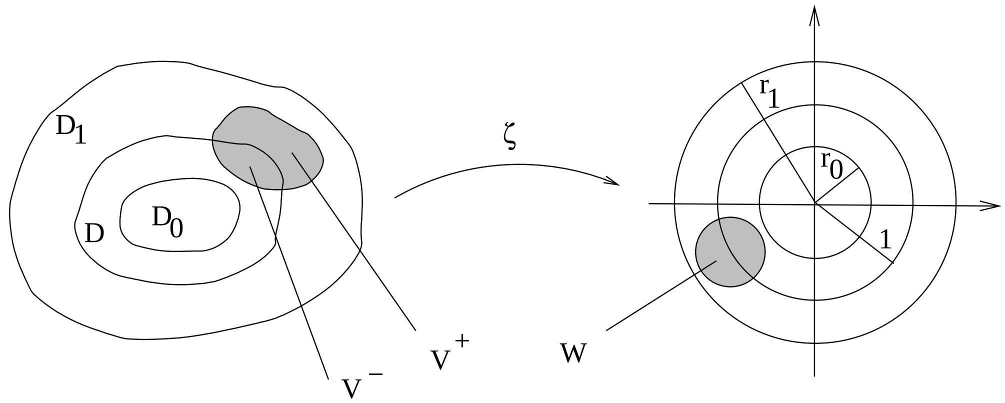


Figure 2.2: Schiffer variation induced coordinate changes

The local coordinate ζ_i on V_i is defined by

$$\zeta_i(z) = \begin{cases} \zeta & \text{for } z \in V_i^+ \\ (v^\epsilon)^{-1} \circ w^\epsilon \circ \zeta(z) & \text{for } z \in V_i^- \end{cases}$$

This is well defined because $v^\epsilon = w^\epsilon$ on $\partial B(0,1)$. It is not hard to see that ζ_i is a homeomorphism.

To see that these charts define a complex structure we need to check that the transition functions are analytic. The new transition functions are the following.

- For $z \in D_1 \backslash D$ the transition function is just the identity.
- $z \in \partial D$ is only covered by the V_i so the transition functions are the identity.
- For $z \in D_0$ there is no transition function as such z are only covered by one chart.
- For $z \in D \backslash D_0$ the transition function is $((v^\epsilon)^{-1} \circ w^\epsilon \circ \zeta) \circ (w^\epsilon \circ \zeta)^{-1} = (v^\epsilon)^{-1}$. We know that v^ϵ is a biholomorphism on $D \backslash D_0$ and therefore so is $(v^\epsilon)^{-1}$.

Remark 2.6.4. It is interesting to note that the information of the Schiffer variation of complex structure is contained in the transition function v^ϵ. Because v^ϵ depends analytically on ϵ it is not surprising that Schiffer variation determines an analytic neighborhood in $T(\Sigma)$.

Remark 2.6.5. In the appropriate domain, $v^\epsilon(z)$ is a biholomorphic function of ϵ and z separately. By Hartogs' theorem (see Theorem 2.2.4), $v^\epsilon(z)$ is in fact a biholomorphic function as a function of two variables. In this particular case the simple formula for $v^\epsilon(z)$ makes this fact easy to check directly.

2.6.2 An atlas for Teichmüller space

Schiffer variation can be used to produce an analytic atlas for Teichmüller space. By Theorem 2.6.3 we know that Schiffer variation gives an local analytic coordinate chart at the base point. Producing charts at other points is simple, but requires the introduction of some more notation. It is important to note in what follows that we are not defining the complex structure for $T(\Sigma)$, but rather defining local coordinates for a pre-existing complex structure.

By Theorem 2.6.3 we can choose a sufficiently small neighborhood N of $0 \in \mathbb{C}$ such that the map $S : N \to T(\Sigma)$ is a biholomorphism.

Definition 2.6.1. For any Riemann surface Σ we define the *Schiffer neighborhood* of the base point $[\Sigma, id, \Sigma] \in T(\Sigma)$ to be

$$\mathcal{N}^\epsilon = S(N) = \{[\Sigma, \nu^\epsilon, \Sigma^\epsilon] \,|\, \epsilon \in N\}.$$

Note that this neighborhood depends on the choices of disks, D_i, where the Schiffer variation is performed, and the corresponding local coordinates ζ_i. These dependencies are hidden in the map S.

We now produce a neighborhood of an arbitrary point $[\Sigma, f, \Sigma_1] \in T(\Sigma)$. Let ζ_i^1 be the local coordinates at the disks D_i on Σ_1. The map $\nu_1^\epsilon : \Sigma_1 \to \Sigma_1^\epsilon$ is given by

$$\nu_1^\epsilon(x) = \begin{cases} x & x \in \Sigma \setminus D \\ (w \circ \zeta_i^1)(x) & x \in D_i \;, i = 1, \ldots, n \;. \end{cases}$$

Let $\mathcal{N}_1^\epsilon$ be the Schiffer neighborhood of the base point in $T(\Sigma_1)$. Since $f : \Sigma \to \Sigma_1$ is a quasiconformal homeomorphism, $f^* : T(\Sigma_1) \to T(\Sigma)$ is a biholomorphism by Theorem 2.3.5. Note that $f^*([\Sigma_1, id, \Sigma_1]) = [\Sigma, f, \Sigma_1]$.

Definition 2.6.2. The *Schiffer neighborhood of* $[\Sigma, f, \Sigma_1]$ is

$$\begin{aligned} \mathcal{N}^\epsilon_{[\Sigma,f,\Sigma_1]} &= f^*(\mathcal{N}_1^\epsilon) \\ &= \{[\Sigma, \nu_1^\epsilon \circ f, \Sigma_1^\epsilon] \,|\, [\Sigma_1, \nu_1^\epsilon, \Sigma_1^\epsilon] \in \mathcal{N}_1^\epsilon\}. \end{aligned}$$

The local coordinate for this neighborhood is $S^{-1} \circ (f^*)^{-1}$ which is analytic by Theorems 2.3.5 and 2.6.3.

To produce an atlas on $T(\Sigma)$ we repeat the above construction to get neighborhoods $\mathcal{N}^{\epsilon}_{[\Sigma,f,\Sigma_1]}$ that cover $T(\Sigma)$. We need to check that the transition functions are analytic. But this is immediate as each local coordinate $S^{-1} \circ (f^*)^{-1}$ is analytic with respect to the canonical complex structure on $T(\Sigma)$.

Remark 2.6.6. It is interesting to note that any equivalence class $[\Sigma, f, \Sigma_1]$ can be represented in the form $[\Sigma, f', \Sigma_1']$ where f' and Σ_1' are constructed by a finite sequence of Schiffer variations. Another way to say this is that any point can be related to the base point by a finite sequence of Schiffer variations.

Chapter 3

Teichmüller Space of Rigged Surfaces

As suggested by its title, this chapter introduces the notion of the Teichmüller space of rigged surfaces. The goal is to prove that this space, and the associated moduli space, are infinite-dimensional complex manifold. Schiffer variation will be used to produce charts for this space. Proving the holomorphicity of the transition functions relies on the universality of the fiber space $V(G)$. Once this is done a simple argument shows that the mapping class group acts fixed-point freely on the rigged Teichmüller space. It follows form this that the moduli space of rigged surfaces is also an infinite-dimensional complex manifold.

3.1 Definitions and notation for rigged Riemann surfaces

An *oriented puncture* on a Riemann surface is a puncture together with an element of $\{+,-\}$. Let Σ be a Riemann surface of finite conformal type (g,n) with ordered and oriented punctures $(p_1,\ldots,p_n)$.

Definition 3.1.1. A *negatively (respectively, positively) oriented local analytic coordinate chart* centered at a negatively (respectively, positively) oriented puncture p_i is a pair (U_i,ϕ_i), where U_i is an open neighborhood of p_i and ϕ_i is an analytic map from U_i to an open neighborhood of 0 in the punctured plane $\mathbb{C}\setminus\{0\}$ (respectively, an open neighborhood of ∞ in $\hat{\mathbb{C}}\setminus\{\infty\}$). See Figure 3.1.

To be brief we often just refer to these as *local coordinates* at the puncture. The reason for considering the "orientation" will become clear when we discuss surfaces with boundary and the sewing operation in Chapter 4. In this chapter the orientation issue can be essentially ignored.

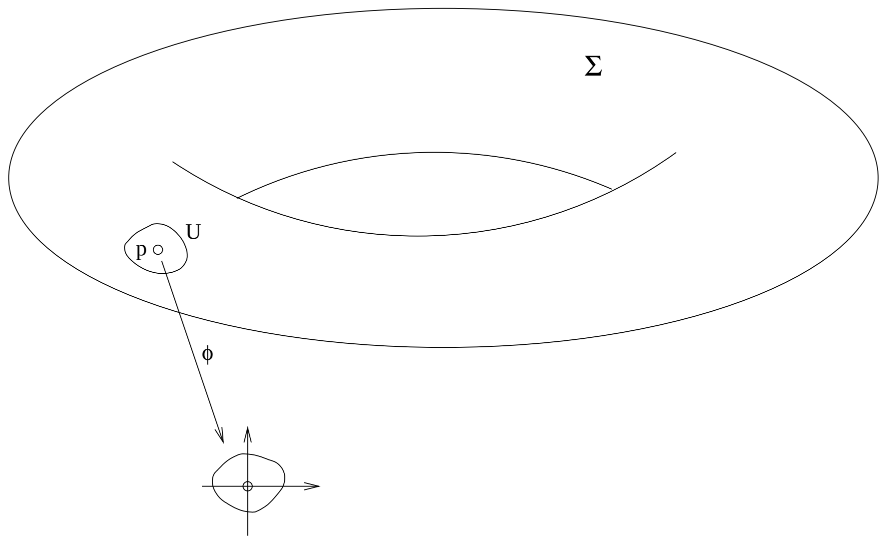


Figure 3.1: Puncture and local coordinate

Remark 3.1.1. If we consider surfaces with distinguished points rather than punctures, then $\phi(p_i) = 0$ or $\phi(p_i) = \infty$ depending on the orientation.

Remark 3.1.2. We can equally well consider the information of the orientation to be carried by the local coordinate rather than by the point.

We call $\phi = (\phi_1, \ldots, \phi_n)$ the local coordinates for Σ, where each ϕ_i has orientation corresponding to that of p_i. It is important to note that ϕ must agree with the order of the punctures in the sense that ϕ_i is the local coordinate at the point labelled p_i. Note that we do not require the neighborhoods U_i to be disjoint.

Definition 3.1.2. The data of a punctured Riemann surface together with local coordinates

$$(\Sigma, (p_1, \ldots, p_n), (\phi_1, \ldots, \phi_n))$$

is called a *rigged Riemann surface.*

Remark 3.1.3. The term *extended surface* is often used for similar data.

We need to describe precisely what the space of all local coordinates is.

Definition 3.1.3. Let $\mathcal{O}_0$ be the complex vector space of all series of the form

$$\sum_{n=1}^{\infty} a_n z^n$$

with $a_1 \neq 0$, that are absolutely convergent in a neighborhood of $0 \in \mathbb{C}$. Similarly, let $\mathcal{O}_\infty$ be the space of series that converge in a neighborhood of $\infty \in \hat{\mathbb{C}}$ and whose first coefficient is non-zero.

Note that $\mathcal{O}_0$ is the space of analytic functions, f, such that $f(0) = 0$ and $f'(0) \neq 0$. Note that $f(1/z) \in \mathcal{O}_\infty$ and $\mathcal{O}_\infty$ can be identified with $\mathcal{O}_0$ in this way. We use $\mathcal{O}$ to stand for either of these spaces when we do not need to distinguish between them.

The space $\mathcal{O}$ has the structure of an (LB)-space. See Huang, ([30], Appendix B) for details.

Let p be a negatively oriented puncture on Σ, and choose a (negatively oriented) local coordinate ϕ. Then any $f \in \mathcal{O}_0$ gives a (negatively oriented) local coordinate $f \circ \phi$ at p.

Definition 3.1.4. For p a negatively oriented puncture on Σ, let

$$\mathcal{O}(p) = \{f \circ \phi \,|\, f \in \mathcal{O}_0\}$$

where ϕ is any (negatively oriented) local coordinate at p.

It is easy to check that $\mathcal{O}(p)$ does not depend on the choice of ϕ and is precisely the space of negatively oriented local coordinates at p.

For a positively oriented puncture, p, we similarly define

$$\mathcal{O}(p) = \{f \circ \phi \,|\, f \in \mathcal{O}_\infty\}$$

and note this is the space of positively oriented local coordinates. We use the same notation in both cases as the information of the orientation is carried by the point p. To be precise we should really write $(p, +)$ and $(p, -)$.

Definition 3.1.5. The *space of all local coordinates*, $\mathcal{O}(\Sigma)$, on $(\Sigma, (p_1, \ldots, p_n))$ is defined by

$$\mathcal{O}(\Sigma) = \mathcal{O}(p_1) \times \cdots \times \mathcal{O}(p_n).$$

It will also be necessary to talk about an atlas for Σ. Let $\{V_\alpha\}_{\alpha \in \mathcal{A}}$ be an open cover for Σ and let $\zeta_\alpha : V_\alpha \to \mathbb{C}$ be the local coordinates. Then $\{(V_\alpha, \zeta_\alpha)\}_{\alpha \in \mathcal{A}}$ is an atlas for Σ. To avoid confusion with local coordinate charts at the punctures we will always use ϕ_i for coordinates at the punctures and ζ_α for coordinates on the surface.

3.2 More about Schiffer variation

As discussed in Section 2.6 Schiffer variation produces a local neighborhood in $T(\Sigma)$. Moreover these neighborhoods can be used to give an analytic atlas for $T(\Sigma)$. Starting from a fixed surface Σ, Schiffer variation on a single disk produces a path $[\Sigma, \nu^\epsilon, \Sigma^\epsilon]$ in $T(\Sigma)$. The surfaces Σ^ϵ are, of course, not the same as the canonical representatives Σ^μ in $V(G)$. In order to view Schiffer variation as occurring in $V(G)$ we must map Σ^ϵ to the canonical representative. Luckily there already exists such a map.

Recall from Section 2.6.1 that $\Sigma_{\mu(\nu^\epsilon)}$ is the topological surface Σ with complex structure pulled back from Σ^ϵ by ν^ϵ. The quasiconformal homeomorphism ν^ϵ, and therefore $\mu(\nu^\epsilon)$, depend analytically on ϵ.

Remark 3.2.1. The notation is a little confusing so let us recap.

- Schiffer variation produces the *varied surface* Σ^ϵ
- Alternatively we can consider the varied surface to be $\Sigma_{\mu(\nu^\epsilon)}$, which is the original surface with a different complex structure.
- $\mu(\nu^\epsilon)$ specifies an equivalence class in $T(G)$. The canonical surface associated with this class is $\Sigma^{\mu(\nu^\epsilon)}$.

The point of all this is to be able to apply Theorem (A.3.5) to get a canonical biholomorphism

$$f^{\mu(\nu^\epsilon)} : \Sigma_{\mu(\nu^\epsilon)} \to \Sigma^{\mu(\nu^\epsilon)}.$$

Using this map we can consider Schiffer variation as giving a family of surfaces $\Sigma^{\mu(\nu^\epsilon)}$.

3.2.1 Schiffer variation as a marked holomorphic family

Refer to Section 2.5.3 for the background on marked holomorphic families.

Schiffer variation produces a family of surfaces Σ^ϵ. It is natural to ask if this family is a marked (n-punctured) holomorphic family. The answer is yes. Since $V(G)$ is universal in the sense of Theorem 2.5.6 we then know there is a morphism of marked holomorphic families from the Schiffer family to $V(G)$. We will explicitly exhibit this map.

We now show that the family of surfaces Σ^ϵ is a holomorphic fiber space over a neighborhood of $0 \in \mathbb{C}$.

Let $d > 0$ be such that for all $|\epsilon| < d$, ϵ is a local coordinate for $T(\Sigma)$ under the Schiffer map S. See Theorem 2.6.3.

Definition 3.2.1. Define the *Schiffer family* $S(\Sigma)$ by

$$S(\Sigma) = \left\{(\epsilon, z) \,|\, |\epsilon| < d, z \in \Sigma_{\mu(\nu^\epsilon)}\right\}.$$

Topologically $S(\Sigma) = B(0,d) \times \Sigma$ because topologically $\Sigma_{\mu(\nu^\epsilon)} = \Sigma$.

This definition is not in Nag [44] and so the following proposition is not formulated. It is an immediate corollary of Theorem 2.6.3.

Proposition 3.2.2. *The Schiffer family $S(\Sigma)$ is a holomorphic fiber space with base space $B(0,d) \subset \mathbb{C}$ and fibers $\Sigma_{\mu(\nu^\epsilon)}$.*

Proof. Using the local coordinates for $\Sigma_{\mu(\nu^\epsilon)}$ described in Section 2.6.1 we will give local coordinates for $S(\Sigma)$. This procedure is trivial in the sense that we just take the original coordinates for the Riemann surface (fiber direction) and the identity on the base space.

- On $\Sigma \setminus D \times B(0,d)$, the charts are

$$U_\alpha \times B(0,d) \longrightarrow \zeta_\alpha(U_\alpha) \times B(0,d)$$

  where $(z,\epsilon) \mapsto (\zeta_\alpha(z), \epsilon)$.

- On $D_0 \times B(0,d)$ the map is

$$(z,\epsilon) \longmapsto (w^\epsilon \circ \zeta(z), \epsilon).$$

- On $V_j^+ \times B(0,d)$,

$$(z,\epsilon) \longmapsto (\zeta(z), \epsilon).$$

- On $V_j^- \times B(0,d)$,

$$(z,\epsilon) \longmapsto ((v^\epsilon)^{-1} \circ w^\epsilon \circ \zeta(z), \epsilon).$$

It has already been shown in Section 2.6.1 that the transition functions are analytic in z. The analyticity in ϵ follows from the fact that $v^\epsilon(z)$ is a biholomorphism in ϵ. As already noted in Remark 2.6.5, $v^\epsilon(z)$ is biholomorphic as a function from $B(0,r_1) \setminus B(0,r_0) \times B(0,d) \subset \mathbb{C}^2 \to \mathbb{C}$.

□

Although we know by the universality of $V(G)$ that there is a morphism of marked families $S(\Sigma) \to V(G)$ we give a direct proof.

Proposition 3.2.3. *There is a morphism of marked (n-punctured) families*

$$(s,F) : S(\Sigma) \longrightarrow V(G)$$

given by

$$(\epsilon, z) \longmapsto ([\mu(\nu^\epsilon)], f^{\mu(\nu^\epsilon)}(z))$$

where $f^{\mu(\nu^\epsilon)}(z) = (w^{\mu(\nu^\epsilon)}(z))_*$.

Proof. We know from Theorem 2.6.3 that $s(\epsilon) = [\mu(\nu^\epsilon)] \in T(G)$ depends holomorphically on ϵ. Actually s is the canonical classifying map h (see Section 2.5.3). From Theorem A.3.5 we know $f^{\mu(\nu^\epsilon)} : \Sigma_{\mu(\nu^\epsilon)} \to \Sigma^{\mu(\nu^\epsilon)}$ is a biholomorphism.

It remains to show that (s,F) is marking preserving. This is equivalent to showing that $(\Sigma, id, \Sigma_{\mu(\nu^\epsilon)})$ is Teichmüller equivalent to $(\Sigma, f^{\mu(\nu^\epsilon)}, \Sigma^{\mu(\nu^\epsilon)})$ via the biholomorphism $f^{\mu(\nu^\epsilon)}$. But this is immediate as the induced map from Σ to Σ is just the identity. □

We could equivalently define the Schiffer family by

$$S'(\Sigma) = \{(\epsilon, z) \,|\, |\epsilon| < d, z \in \Sigma^\epsilon)]\}\,.$$

and it is easy to see that this is isomorphic, as a marked family of surfaces, to $S(\Sigma)$. The marked triples $(\Sigma, id, \Sigma_{\mu(\nu^\epsilon)})$ are equivalent to $(\Sigma, \nu^\epsilon, \Sigma^\epsilon)$ via the biholomorphism $\nu^\epsilon : \Sigma_{\mu(\nu^\epsilon)} \to \Sigma^\epsilon)$. In the future we will refer to either of these as $S(\Sigma)$.

For convenience we have been working in a neighborhood of the base point in $T(\Sigma)$. More generally, the following holds.

Corollary 3.2.4. *Given* $[\Sigma, g, \Sigma_1]$ *there is a morphism of marked holomorphic families*

$$S(\Sigma_1) \longrightarrow V(G)$$

where $S(\Sigma_1)$ *is the Schiffer family corresponding to* $[\Sigma, g \circ \nu_1^{\epsilon}, \Sigma_1^{\epsilon}]$. *See Section 2.6.2.*

Proof. The above proof of Proposition 3.2.3 be copied almost without change. ☐

3.3 Teichmüller space of rigged surfaces

Let (Σ, α) be a Riemann surface with (oriented) punctures ordered by α.

Definition 3.3.1. A *marked Riemann surface with punctures and local coordinates* modelled on (Σ, α) is a quadruple $(\Sigma, f, \Sigma_1, \phi)$, where Σ_1 is a Riemann surface, $f : \Sigma \to \Sigma_1$ is a quasiconformal homeomorphism called the marking map and $\phi = (\phi_1, \ldots, \phi_n) \in \mathcal{O}(\Sigma_1)$ are the local coordinates at the punctures $(p_1, \ldots, p_n)$ on Σ_1 whose order is induced from α by f.

Definition 3.3.2. Let $\bar{M}(\Sigma)$ be the collection of all marked Riemann surface with punctures and local coordinates $(\Sigma, f, \Sigma_1, \phi)$, where $\phi \in \mathcal{O}(\Sigma_1)$.

Definition 3.3.3. Two marked Riemann surfaces punctures with local coordinates $(\Sigma, f, \Sigma_1, \phi)$ and $(\Sigma, g, \Sigma_2, \psi) \in \bar{M}(\Sigma)$ are called *(rigged) Teichmüller equivalent* $(\sim_B)$ if and only if $(\Sigma, f, \Sigma_1) \sim (\Sigma, g, \Sigma_2)$ via a biholomorphism σ such that $\phi_i = \psi_i \circ \sigma$ for $i = 1, \ldots, n$.

To emphasize the role played by the ordering we note that $\phi_i = \psi_i \circ \sigma$ makes sense only when σ is order preserving (see the discussion after Definition 2.3.3).

The following simple example may help to clarify the definition of (rigged) Teichmüller equivalence. Pick $\sigma \in \mathrm{Aut}(\Sigma)$ that does not permute the punctures. Then $[\Sigma, id, \Sigma] = [\Sigma, \sigma, \Sigma]$ whereas $[\Sigma, id, \Sigma, \phi] \neq [\Sigma, \sigma, \Sigma, \phi]$ for any ϕ, unless $\sigma = id$. However, it is the case that $[\Sigma, id, \Sigma, \phi] = [\Sigma, \sigma, \Sigma, \phi \circ \sigma^{-1}]$. Along the same lines we have the following simple statement.

Proposition 3.3.1. $[\Sigma, f, \Sigma_1, \phi] = [\Sigma, f, \Sigma_1, \psi] \Longleftrightarrow \phi = \psi$.

Proof. If $[\Sigma, f, \Sigma_1, \phi] = [\Sigma, f, \Sigma_1, \psi]$, then by definition of the equivalence relation there exists $\sigma \in \mathrm{Aut}(\Sigma_1)$ such that $f \circ \sigma \circ f^{-1}$ is homotopic to the identity. This implies that σ is homotopic to the identity and thus by Proposition 2.2.7, $\sigma = id$. Therefore $\phi = \psi \circ \sigma = \phi$. The implication in the other direction is immediate. □

Definition 3.3.4. The *rigged Teichmüller space*, $\widetilde{T}(\Sigma)$, of (Σ, α) is $\tilde{M}/\sim_B$. The equivalence classes are denoted $[\cdot,\cdot,\cdot,\cdot]$.

The space $\widetilde{T}(\Sigma)$ will be used to obtain the moduli space of rigged surfaces by quotienting by the modular group. Therefore the structure of $\widetilde{T}(\Sigma)$ needs to be understood. This requires substantial work and the final results appear in Section 3.3.3.

The Teichmüller space of Riemann surfaces with punctures and local coordinates has been defined directly in terms of Riemann surfaces in Definition 3.3.4. To use the isomorphism between $T(\Sigma)$ and $T(G)$ to our advantage we need to define everything purely in terms of the Fuchsian groups.

3.3.1 Formulation in terms of $T(G)$

Remark 3.3.2. This section is not logically needed in the results of this thesis. However, it may be conceptually helpful at times.

The are several ways to prescribe the data of a Riemann surface with punctures and local coordinates.

- A closed Riemann surface $\hat{\Sigma}$ of genus, g with distinguished points $p_1, \ldots, p_n$ and local coordinates ϕ_i at the points p_i. It is important to note that in the prescription of the 'punctures', p_i, an order has implicitly been given.

- A Riemann surface, Σ, of finite conformal type (g, n). An ordering of the punctures, which is a function α from the set $\{1, \ldots, n\}$ to the set of punctures. Local coordinates ϕ_i at the punctures p_i.

- A torsion-free, finitely generated, first-kind Fuchsian group G. An ordering, α, of the punctures, and a choice of local coordinates ϕ_i at the punctures p_i.

To see the equivalence between (1) and (2) it is enough to note that

$$\Sigma = \hat{\Sigma} \backslash \{p_1, \ldots, p_n\}.$$

The equivalence with (3) is guaranteed by proposition (A.3.1) where G is chosen such that $\Sigma = U/G$.

Recall Definition 3.3.4 of the Teichmüller space, $\widetilde{T}(\Sigma)$, of Riemann surfaces with punctures and local coordinates. We will give a reformulation of this definition in terms of the Teichmüller group $T(G)$ and prove the equivalence of the two definitions. See Section A.3 for background material.

Here we need to use the result of Teichmüller's theorem (see Section A.3.4) that there exists a unique (extremal) representative μ_T of each equivalence class $[\mu] \in T(G)$. Given such a μ_T we have the unique representative marked triple $(\Sigma, f^{\mu_T}, \Sigma^\mu)$ of the equivalence class $[\Sigma, f^\mu, \Sigma^\mu] \in T(\Sigma)$. Recall that $f^{\mu_T} = (w^{\mu_T})_*$, where $(w^\mu)_*$ is the canonical quasiconformal map $\Sigma = U/G \to \Sigma^\mu = w^\mu(U)/G^\mu$. See Theorem A.3.5 and its proof for details.

Definition 3.3.5. The *rigged Teichmüller group*, $\widetilde{T}(G)$ of (g, α) is defined by

$$\widetilde{T}(G) = \{([\mu], (\phi_1, \ldots, \phi_n)) \,|\, \mu \in T(G) \text{ and } \phi_i \text{ are the local coordinate on } \Sigma^\mu\}$$

where α is an ordering of the punctures on $\Sigma = U/G$ and the ordering is pushed forward to the surfaces Σ^μ via f^{μ_T}.

Proposition 3.3.3. *There is a canonical isomorphism*

$$\tilde{\rho} : \widetilde{T}(G) \longrightarrow \widetilde{T}(\Sigma)$$

given by

$$([\mu], (\phi_1, \ldots \phi_n)) \longmapsto [\Sigma, f^{\mu_T}, \Sigma^\mu, (\phi_1, \ldots, \phi_n)].$$

Proof. The proposition follows from Corollary A.3.9. Once again, the fundamental idea is that w^{μ_T} and Σ^μ depend only on the equivalence class of μ.

The inverse map to ρ needs a little thought. Given $[\Sigma, g, \Sigma_1, (\psi_1, \ldots, \psi_n)]$ we need to produce a μ in $L^\infty(U, G)_1$ and produce local coordinates on the canonical surface Σ^μ. A little work must be done as μ_g may not be in $L^\infty(U, G)_1$. These considerations are taken care of in Theorem A.3.6 where the lift of g is suitably normalized.

Corollary A.3.9 guarantees such a μ exists and we denote it by $\tilde{\mu}_g$. From this we also see that

$$[\Sigma, g, \Sigma_1] = [\Sigma, f^{\tilde{\mu}_g}, \Sigma^{\tilde{\mu}_g}] = [\Sigma, g, \Sigma_1]$$

and in particular this means there is a biholomorphism $\sigma_g = f^{\tilde{\mu}_g} \circ g^{-1} : \Sigma_1 \to \Sigma^{\tilde{\mu}_g}$. Using this biholomorphism, the local coordinates ψ_i on Σ_1 can be translated to local coordinates $\psi_i \circ \sigma_g^{-1}$.

So from $[\Sigma, g, \Sigma_1, (\psi_1, \ldots, \psi_n)]$ we have produced an element $([\tilde{\mu}_g], (\psi_1 \circ \sigma_g^{-1}, \ldots, \psi_i \circ \sigma_g^{-1}))$ in $\widetilde{T}(G)$. This assignment is the inverse to ρ because

$$[\Sigma, f^{\tilde{\mu}_g}, \Sigma^{\tilde{\mu}_g}, (\psi_1 \circ \sigma_g^{-1}, \ldots, \psi_i \circ \sigma_g^{-1})] = [\Sigma, g, \Sigma_1, (\psi_1, \ldots, \psi_n)]$$

via σ_g. □

3.3.2 Local coordinates on holomorphic families

The motivation of this section is that we want to be able to compare local coordinates on marked holomorphic families of Riemann surfaces. In general if $f_t : \Sigma \to \Sigma_t$ is a marked holomorphic family of surfaces with $\Sigma_0 = \Sigma$, then we would like to be able to compare the local coordinates on Σ with those on Σ_t. That is, we want a way to identify $\mathcal{O}(\Sigma)$ with $\mathcal{O}(\Sigma_t)$.

Remark 3.3.4. To be more precise, it is the fibers Σ_t over t that form a holomorphic family and f_t are the marking maps.

For our later purposes, in particular proving that $\widetilde{T}(\Sigma)$ is a complex manifold, it is enough to restrict to the special case of Schiffer variation.

Recall from Section 2.6 and Subsection 3.2.1 that Schiffer variations produces a holomorphic family $[\Sigma, \nu^\epsilon, \Sigma^\epsilon]$. In this case ν^ϵ in the identity away from the disks where the Schiffer variation is performed. By choosing the disks away from the punctures, ν^ϵ will be the identity in a neighborhood of the punctures.

Thus the local coordinates on Σ^ϵ can be naturally identified with the local coordinates on Σ. In particular if ϕ is a local coordinate on Σ^ϵ then $\phi \circ \nu^\epsilon = \phi$ is a local coordinate on Σ.

Remark 3.3.5. If we map this family to the universal family $V(G)$ the marking maps are holomorphic in a neighborhood of the punctures. This is because $\mu(\nu^\epsilon)$ is zero in these neighborhoods and so $f^{\mu(g^\epsilon)} : \Sigma \to \Sigma^{\mu(g^\epsilon)}$ is holomorphic. In this case we can still use the marking map $f^{\mu(g^\epsilon)}$ to identify the local coordinates.

The rest of this section is aimed at formulating a conjecture and giving a partial proof. If the conjecture is true then the proofs of some later results could be simplified.

By the universality of $V(G)$ it is enough to consider marked families of the form $[\Sigma, f^{\mu_t}, \Sigma^{\mu_t}]$ where μ_t depends holomorphically on t.

For simplicity, let p be the only puncture on Σ. Let D be a neighborhood of p, and let h be an analytic map from D to the punctured unit disk.

Remark 3.3.6. Using this h we can map $\mathcal{O}(p)$ to $\mathcal{O}$ by $\phi \mapsto \phi \circ h^{-1}$.

Let $\gamma = h^{-1}(S^1)$ and define a curve around the puncture on Σ^{μ_t} by $\gamma^{\mu_t} = f^{\mu_t}(\gamma)$. Let h^{μ_t} be an analytic map from the interior of γ^μ to the unit disk normalized such that $(h^{\mu_t} \circ f^{\mu_t} \circ h^{-1})(1) = 1$.

Conjecture 3.3.7. The map h^{μ_t} is unique and depends on t holomorphically in the sense that

$$h^\nu \circ f^\nu \circ h^{-1} : B(0,1) \setminus \{0\} \longrightarrow B(0,1) \setminus \{0\}$$

is holomorphic.

Remark 3.3.8. The following (indirect) argument only proves real analyticity.

The uniqueness is clear because the identity is the only conformal map from $B(0,1)$ to itself fixing 0 and 1. The holomorphicity really comes from the fact the w^μ, and hence f^ν, is holomorphic in μ. We need a little work to see this. For simplicity and to avoid a notation conflict we now change from μ_t to ν_t. That is we use f^{ν_t}, etc. Let

$$k^{\nu_t} = h^{\nu_t} \circ f^{\nu_t} \circ h^{-1} : B(0,1) \setminus \{0\} \longrightarrow B(0,1) \setminus \{0\}$$

We must show that $k^{\nu_t}(z_0)$ depends analytically on t for fixed z_0. Fix $z_0 \in B(0,1) \setminus \{0\}$.

Recall that $f^\nu = (w^\mu)_*$ and that coordinate charts on $\Sigma^\mu = w^\mu / G^\mu$ are given by local inverses of the projection π^μ. By choosing these local inverses appropriately we can get coordinates charts ζ about $h^{-1}(z_0)$ and ζ^ν about $f^\nu(h^{-1}(z_0))$ such that $\zeta^\nu \circ f^\nu \circ \zeta^{-1} = w^\nu$.

In a neighborhood of z_0 we can now write k^{ν_t} as

$$k^{\nu_t} = \left(h^{\nu_t} \circ (\zeta^{\nu_t})^{-1}\right) \circ \left(\zeta^{\nu_t} \circ f^{\nu_t} \circ \zeta^{-1}\right) \circ \left(\zeta \circ h^{-1}\right).$$

Note that each expression in parentheses is a function on a neighborhood in $\mathbb{C}$. By the above choice of coordinates $\zeta^\nu \circ f^\nu \circ \zeta^{-1} = w^\nu$. To simplify notation, let $g = \zeta \circ h^{-1}$. The first term $h^{\nu_t} \circ (\zeta^{\nu_t})^{-1}$ is analytic, so the complex dilation of k^ν is

$$\begin{aligned} \mu_{k^{\nu_t}} &= \mu_{w^{\nu_t} \circ g} \\ &= \frac{\mu_g + (\mu_{w^{\nu_t}} \circ g)\,(r_g)}{1 + (\mu_{w^{\nu_t} \circ f})\, r_g \mu_g} \\ &= (\mu_{w^{\nu_t}} \circ g)\, r_g \end{aligned}$$

Where we have used the fact that $\mu_g = 0$ and Formula A.2. Because $\mu_{w^{\nu_t}} = \nu_t$ we have

$$\mu_{k^{\nu_t}}(z) = \nu_t(g(z)) \frac{\overline{g_z}}{g_z} \tag{3.1}$$

for z in a neighborhood of z_0, and so $\mu_{k^{\nu_t}}(z)$ depends analytically on t. We can repeat this argument for any z_0 and therefore $\mu_{k^\nu}(z)$ depends analytically on t for any $z \in B(0,1) \setminus \{0\}$.

By conformal mapping we can think of $k^\nu(z)$ a quasiconformal mapping from the upper half-plane to itself and thus $\mu_{k^{\nu_t}}(z) \in L^\infty(U)_1$.

Because $k^{\nu_t}(z)$ fixes $0, 1$ and ∞, it is the normalized solution to the Beltrami equation with Beltrami coefficient given by Equation 3.1. Therefore $k^{\nu_t}(z)$ depends real analytically on t. (Actually $k^\nu(z) = w_\nu$. See Nag [44, page 43]).

3.3.3 Complex manifold structure of rigged Teichmüller space

Our aim is to produce a complex manifold (and also bundle) structure on $\widetilde{T}(\Sigma)$. It has been defined with a fiber structure. The fiber over a point $[\Sigma, f, \Sigma_1]$ is the space of local coordinates $\mathcal{O}(\Sigma_1)$. Locally we need to produce a trivialization of these fibers. That is, a map from $\mathcal{O}(\Sigma_1)$ to $\mathcal{O}$.

As in Section 3.3.2 we work with neighborhoods produced by Schiffer variation, and these can easily be trivialized. As described in Section 2.6.2, such neighborhoods cover $T(\Sigma)$.

If two such neighborhoods intersect then we must show that the transition function is holomorphic. To compare these we consider the marked holomorphic families produced by Schiffer variation and map each one biholomorphically into the universal space $V(G)$ via maps H_1 and H_2. Then $H_2^{-1} \circ H_1$ is a biholomorphic map between the two marked families. It follows that the transition function $\mathcal{O} \to \mathcal{O}$ between the local coordinates is holomorphic.

Theorem 3.3.9. *The rigged Teichmüller space $\widetilde{T}(\Sigma)$ is a complex manifold and more over has a bundle structure, with base space $T(\Sigma)$ and fiber model the infinite-dimension space $\mathcal{O}$.*

Proof. Recall from Section 2.6.2 that Schiffer variation can be used to produce an atlas for the complex manifold $T(\Sigma)$

We define two Schiffer neighborhoods simultaneously indexed by $i = 1, 2$. Let $[\Sigma, g_i, \Sigma_i] \in T(\Sigma)$. By performing Schiffer variation on Σ_i with parameter $\epsilon_i \in N_i \subset \mathbb{C}$ we get neighborhoods

$$\mathcal{N}_i = \{[\Sigma, \nu_i^{\epsilon_i} \circ g_i, \Sigma^{\epsilon_i}] \,|\, \epsilon_i \in N_i\}.$$

of $[\Sigma, g_i, \Sigma_i]$. Recall that from Theorem 2.6.3 the map $\epsilon_i \mapsto [\Sigma, \nu^{\epsilon} \circ g_i, \Sigma_i^{\epsilon}]$ is holomorphic.

We also have a marked holomorphic families of surfaces $\Sigma_i^{\epsilon_i}$ over N_i, with marking map $\nu_i^{\epsilon_i} \circ g_i : \Sigma \to \Sigma_i^{\epsilon}$. Denote these marked families by $S(\Sigma_i)$ as in Section 3.2.1.

For simplicity assume that Σ has one puncture and choose local coordinates h_i at the punctures on Σ_i. These choices will determine the trivializations.

We can choose to perform the Schiffer variation away from the puncture and so ν_i^ϵ is the identity in a neighborhood of the puncture. On the surfaces $\Sigma_i^{\epsilon_i}$ we can thus choose the local coordinate to be h_i also.

From the neighborhoods $\mathcal{N}_i$ in $T(\Sigma)$ we get the sets

$$\widetilde{\mathcal{N}_i} = \{[\Sigma, \nu_i^{\epsilon_i} \circ g_i, \Sigma_i^{\epsilon_i}, \phi_i] \,|\, \epsilon_i \in N_i \text{ and } \phi_i \in \mathcal{O}(\Sigma_i^{\epsilon_i})\}.$$

in $\widetilde{T}(\Sigma)$. All we are doing is adding the fibers $\mathcal{O}(\Sigma_i^{\epsilon_i})$ to the points in $\mathcal{N}_i$.

We define the trivializations

$$p_i : \widetilde{\mathcal{N}_i} \longrightarrow T(\Sigma) \times \mathcal{O}$$

by

$$[\Sigma, \nu_i^{\epsilon_i} \circ g_i, \Sigma^\epsilon, \phi_i] \longmapsto [\Sigma, \nu_i^{\epsilon_i} \circ g_i, \Sigma^\epsilon], \phi_i \circ h_i^{-1}.$$

Assume $\mathcal{N}_1 \cap \mathcal{N}_2 \neq \emptyset$. This means that there are some values of ϵ_1 and ϵ_2 such that $[\Sigma, \nu_i^{\epsilon_1} \circ g_1, \Sigma_1^{\epsilon_1}] = [\Sigma, \nu_2^{\epsilon_2} \circ g_2, \Sigma_2^{\epsilon_2}]$. Let $\sigma_{\epsilon_1, \epsilon_2} : \Sigma_1^{\epsilon_1} \to \Sigma_2^{\epsilon_2}$ be the biholomorphism that realizes this equality. Note we do not assume that $[\Sigma, g_1, \Sigma_1] = [\Sigma, g_2, \Sigma_2]$.

The aim is to prove that the transition function $p_2 \circ (p_1)^{-1}$ is holomorphic. As ϵ_1 varies, ϵ_2 must vary to maintain the equality $[\Sigma, \nu_i^{\epsilon_1} \circ g_1, \Sigma_1^{\epsilon_1}] = [\Sigma, \nu_2^{\epsilon_2} \circ g_2, \Sigma_2^{\epsilon_2}]$. Because the Schiffer maps $N \to T(\Sigma) \leftarrow M$ are holomorphic ϵ_2 depends on ϵ_1 holomorphically (on the appropriately restricted domain). We express this dependence by writing $\epsilon_2 = s(\epsilon_1)$.

When $[\Sigma, \nu_i^{\epsilon_1} \circ g_1, \Sigma_1^{\epsilon_1}, \phi_1] = [\Sigma, \nu_2^{\epsilon_2} \circ g_2, \Sigma_2^{\epsilon_2}, \phi_2]$ then by definition $\phi_2 = \phi_1 \circ \sigma_{\epsilon_1, \epsilon_2}^{-1}$. Since ϵ_2 depends on ϵ_1 the notation can be simplified to σ_{ϵ_1}.

We can now write down the transition function. The map from $T(\Sigma) \to T(\Sigma)$ is the identity map, So only the map $\mathcal{O} \to \mathcal{O}$ needs to be investigated.

Let $f \in \mathcal{O}$. Then

$$\begin{aligned} p_1^{-1}([\Sigma, \nu_1^{\epsilon_1} \circ g_1, \Sigma^{\epsilon_1}], f) &= [\Sigma, \nu_1^{\epsilon_1} \circ g_1, \Sigma^{\epsilon_1}, f \circ h_1] \\ &= [\Sigma, \nu_2^{s(\epsilon_1)} \circ g_2, \Sigma^{s(\epsilon_1)}, (f \circ h_1) \circ \sigma_{\epsilon_1}^{-1}]. \end{aligned}$$

Now p_2 maps this element to

$$[\Sigma, \nu_2^{s(\epsilon_1)} \circ g_2, \Sigma^{s(\epsilon_1)}], (f \circ h_1) \circ \sigma_{\epsilon_1}^{-1} \circ h_2^{-1}.$$

Thus the transition function $\mathcal{O} \to \mathcal{O}$ is given by

$$f \mapsto (f \circ h_1) \circ \sigma_{\epsilon_1}^{-1} \circ h_2^{-1}$$

which depends on ϵ_1 only through σ_{ϵ_1}. So the problem has been reduced to showing that σ_{ϵ_1} depends holomorphically on ϵ in the following sense.

The maps σ_{ϵ_1} can be used to produce a map of marked families

$$(\sigma, s) : S(\Sigma_1) \longrightarrow S(\Sigma_2)$$

defined by

$$(z, \epsilon_1) \longmapsto (\sigma_{\epsilon_1}(z), s(\epsilon_1)).$$

Remark 3.3.10. Actually the domain of σ is not all of $S(\Sigma_1)$ as we are working on the intersection of the two neighborhoods. It should be clear what is meant with this abuse of notation.

Saying that σ_{ϵ_1} depends holomorphically on ϵ means that (σ, s) is a morphism of marked families. In particular σ is a holomorphic map between complex manifolds. (See Definition 2.5.6).

By mapping into the universal family $V(G)$ we will be able to express σ as the composition of morphisms of marked families.

By the universality of $V(G)$, Theorem 2.5.6, there exists morphisms of marked families

$$(H_i, h_i) : S(\Sigma_i) \longrightarrow V(G).$$

In this case the classifying maps h_i are the maps

$$\epsilon_i \longmapsto [\Sigma, \nu_i^{\epsilon_i} \circ g_i, \Sigma^{\epsilon_i}]$$

or rather $\epsilon_i \mapsto [\mu(\nu_i^{\epsilon_i} \circ g_i)]$ to be precise. Note that the maps s, by definition, is the composition $h_2 \circ h_1^{-1}$.

Remark 3.3.11. We can be more concrete. The neighborhood of $V(G)$ is formed by the surfaces $\Sigma^{\mu(\nu^\epsilon)}$ and the marking map is $f^{\mu(\nu^\epsilon)} : \Sigma \to \Sigma^{\mu(\nu^\epsilon)}$ For fixed ϵ the biholomorphism H between the surfaces in these families is $f^{\mu(\nu^\epsilon)} \circ \nu^\epsilon : \Sigma^\epsilon \to \Sigma^{\mu(\nu^\epsilon)}$.

The map

$$(H_2 \circ H_1^{-1}, h_2 \circ h_1^{-1}) : S(\Sigma_1) \longrightarrow S(\Sigma_2)$$

is a morphism of marked families. In particular, for fixed ϵ_1, the biholomorphism

$$H_2 \circ H_1^{-1} : \Sigma_1^{\epsilon_1} \to \Sigma_2^{s(\epsilon_1)}$$

realizes the equality $[\Sigma, \nu_i^{\epsilon_1} \circ g_1, \Sigma_1^{\epsilon_1}] = [\Sigma, \nu_2^{\epsilon_2} \circ g_2, \Sigma_2^{\epsilon_2}]$. This follows from the fact that by definition, a morphism of marked (n-pointed) families is marking preserving.

Such a biholomorphism is unique because there are no homotopically non-trivial automorphisms or a Riemann surface with $2g - 2 + n > 0$. (See comments after Definition 2.3.3). Therefore $H_2 \circ H_1^{-1} = \sigma_{\epsilon_1}$. This implies $(\sigma, s) = (H_2 \circ H_1^{-1}, h_2 \circ h_1^{-1})$ and so (σ, s) is a morphism of marked families. In particular, this means σ_{ϵ_1} depends holomorphically on ϵ_1.

Thus we have concluded that the transition function

$$f \mapsto (f \circ h_1) \circ \sigma_{\epsilon_1}^{-1} \circ h_2^{-1}$$

is holomorphic in ϵ and so $\widetilde{T}(\Sigma)$ is a complex manifold.

By the construction of the trivializations we have automatically given a bundle structure with fiber model $\mathcal{O}$.

□

In the complex structure defined above the trivializations depended on a choice of local coordinate h for each chart. This choice does not affect the complex structure as choosing a different h would lead to a transition function that is just the composition with a fixed map. This is holomorphic with respect to the original complex structure.

Remark 3.3.12. Let $\widetilde{V}(G)$ be defined by taking $V(G)$ and placing the fiber $\mathcal{O}(\Sigma_\mu)$ over each point $[\mu]$. Our construction of the bundle structure of $\widetilde{T}(\Sigma)$ above essentially shows that $\widetilde{V}(G)$ is also a holomorphic bundle. One expects the projections

$$\widetilde{V}(G) \longrightarrow V(G) \longrightarrow T(G).$$

to be holomorphic.

3.4 The mapping class group and rigged moduli space

The definition of the mapping class group and its action on Teichmüller space is discussed in Section 2.3.2 In this section we will show that the mapping class group acts fixed-point freely on rigged Teichmüller space. We thus conclude that the moduli space of rigged surface is a complex manifold

Definition 3.4.1. We define the action of $\mathrm{PMod}(\Sigma)$ on $\widetilde{T}(\Sigma)$ by its usual action on $T(\Sigma)$, that is,

$$\rho \cdot [\Sigma, f, \Sigma_1, \phi] = [\Sigma, f \circ \rho, \Sigma_1, \phi]$$

where $\rho \in \mathrm{PMod}(\Sigma)$.

Note that by definition of the mapping class group, ρ preserves the order of the punctures and so $f \circ \rho$ induces the same order on Σ_1 as f. If this were not the case, the local coordinates $\phi = (\phi_1, \ldots, \phi_n)$ would not make sense in $[\Sigma, f \circ \rho, \Sigma_1, \phi]$.

We need to check that the action is well defined. If $[\Sigma, f, \Sigma_1, \phi] = [\Sigma, g, \Sigma_2, \psi]$, then by definition there exists $\sigma : \Sigma_1 \to \Sigma_2$ such that $g^{-1} \circ \sigma \circ f$ is homotopic to the identity. Then for any $\rho \in \mathrm{PMod}(\Sigma)$, $(g \circ \rho)^{-1} \circ \sigma \circ (f \circ \rho) = \rho^{-1} \circ (g^{-1} \circ \sigma \circ f) \circ \rho$ is homotopic to the identity. Thus $\rho \cdot [\Sigma, f, \Sigma_1, \phi] = \rho \cdot [\Sigma, g, \Sigma_2, \psi]$

Definition 3.4.2. The *rigged moduli space* $\widetilde{\mathcal{M}}(\Sigma)$ of Riemann surfaces with punctures and local coordinates modelled on (Σ, α) is defined to be $\bar{M}(\Sigma)/\sim_R$, where $(\Sigma, f, \Sigma_1, \phi) \sim_R (\Sigma, g, \Sigma_2, \psi)$ if and only if there exists a biholomorphism $\sigma : \Sigma_1 \to \Sigma_2$ such that $\phi_i = \psi_i \circ \sigma$, for $i = 1, \ldots, n$.

Remark 3.4.1. Recall that the order of the punctures on Σ_1 and Σ_2 is induced by f and g. So the condition $\phi_i = \psi_i \circ \sigma$ implies that σ preserves the order of the punctures.

Just as in the standard case it is easy to show that $\widetilde{\mathcal{M}}(\Sigma)$ is isomorphic to the space of conformal equivalence classes of *rigged Riemann surfaces.*

Proposition 3.4.2. *The boundary Teichmüller space divided by the action of the mapping class group gives the moduli space. That is, the map*

$$T : \widetilde{\mathcal{M}}(\Sigma) \longrightarrow \widetilde{T}(\Sigma)/\mathrm{PMod}(\Sigma)$$

given by

$$(\Sigma, f, \Sigma_1, \phi) \longmapsto [\Sigma, f, \Sigma_1, \phi]$$

is a bijection.

Proof. Well-Defined: If $(\Sigma, f, \Sigma_1, \phi) \sim_R (\Sigma, g, \Sigma_2, \psi)$ then we must produce a $\rho \in \mathrm{PMod}(\Sigma)$ such that $\rho \cdot [\Sigma, g, \Sigma_2, \psi] = [\Sigma, f, \Sigma_1, \phi]$. By the definition of $\sim_R$ there exists a biholomorphism $\sigma : \Sigma_1 \to \Sigma_2$ such that $\phi = \psi \circ \sigma$. Let $\rho = g^{-1} \circ \sigma \circ f$. As emphasized in Remark 3.4.1, the definition of $\sim_R$ implies that $\rho \in \mathrm{PMod}(\Sigma)$. Then $\rho \cdot [\Sigma, g, \Sigma_2, \psi] = [\Sigma, g \circ (g^{-1} \circ \sigma \circ f), \Sigma_2, \psi] = [\Sigma, \sigma \circ f, \Sigma_2, \psi]$. By using the same σ we see that $[\Sigma, \sigma \circ f, \Sigma_2, \psi]$ is Teichmüller equivalent to $[\Sigma, f, \Sigma_1, \phi]$. To check this we note that $(\sigma \circ f)^{-1} \circ \sigma \circ f = Id$ and $\phi = \psi \circ \sigma$.

Injective: If $\rho \cdot [\Sigma, g, \Sigma_2, \psi] = [\Sigma, f, \Sigma_1, \phi]$ then we must show that $(\Sigma, g, \Sigma_2, \psi) \sim_R (\Sigma, f, \Sigma_1, \phi)$. By definition $\rho \cdot [\Sigma, g, \Sigma_2, \psi] = [\Sigma, g \circ \rho, \Sigma_2, \psi]$ and thus there exists a biholomorphism $\sigma : \Sigma_1 \to \Sigma_2$ such that $\phi = \psi \circ \sigma$. Using σ we see that $(\Sigma, g, \Sigma_2, \psi) \sim_R (\Sigma, f, \Sigma_1, \phi)$

Onto: Clear. □

In the standard case of punctured Riemann surfaces, the action of the mapping class group is not fixed point free. The canonical example is that of a hyperelliptic involution with punctures at the fixed points. Such automorphisms can not fix the local coordinates at the punctures, so in the case of rigged surfaces we expect the action to be fixed point free. This is even the case for surfaces with punctures and associated tangent vectors. See for example Bakalov and Kirillov [4, Chapter 6].

Lemma 3.4.3. *The action of* $\mathrm{PMod}(\Sigma)$ *on* $\widetilde{T}(\Sigma)$ *is fixed-point-free.*

Proof. Let $\rho \in \mathrm{PMod}(\Sigma)$ and assume that ρ fixes an element in $\widetilde{T}(\Sigma)$. That is,

$$\rho \cdot [\Sigma, f, \Sigma_1, \phi] = [\Sigma, f \circ \rho, \Sigma_1, \phi] = [\Sigma, f, \Sigma_1, \phi]$$

Then by the definition of equivalence, there exists a biholomorphism $\sigma : \Sigma_1 \to \Sigma_1$ such that $\phi = \phi \circ \sigma$, which implies that σ is the identity in a neighborhood of the punctures. Since σ is a biholomorphism and equal to the identity on an open set it bust be the case that $\sigma \equiv Id$. Also, by assumption we know that $(f \circ \rho)^{-1} \circ \sigma \circ f$ is homotopic to the identity. As $\sigma \equiv Id$, it follows that $(f \circ \rho)^{-1} \circ \sigma \circ f = \rho^{-1}$ and so ρ is homotopic to the identity. That is, $\rho \in Q_0(\Sigma)$ which implies that $\rho = Id \in \mathrm{PMod}(\Sigma)$.

Since ρ and $[\Sigma, f, \Sigma_1, \phi]$ were arbitrary we have shown that if ρ fixes any element then $\rho = Id$. Thus the action is fixed-point-free. □

From Theorem 2.3.7 we already know that the mapping class group acts properly discontinuously. Combining this with our previous theorem it follows that the moduli space inherits whatever manifold structure Teichmüller space has. In particular we have the following important result.

Theorem 3.4.4. *$\widetilde{\mathcal{M}}(\Sigma)$ is a complex manifold.*

3.5 More on marked families of Riemann surfaces

Remark 3.5.1. Although the construction in this section is useful it is not strictly needed in the development of this thesis. However, this section is referred to for simpler alternative proofs of some results.

We give a construction of a marked family from an analytic family of Beltrami differentials. Recall from Section 2.5.3 the definition of a marked family of Riemann surfaces.

Let Σ be a surface of type of conformal type(g, n) (with $2g - 2 + n > 0$ as always) and let B be a domain in $\mathbb{C}$. For $t \in B$, let Σ_t be a family of Riemann surfaces and

$$g_t : \Sigma \to \Sigma_t$$

be a family of quasiconformal maps such that

$$\mu_t = \mu_{g_t} \in L^{\infty}_{(-1,1)}(\Sigma)$$

depends analytically on t. Let

$$\mathcal{F} = \{(t, z) \,|\, t \in B, z \in \Sigma_t\}$$

Proposition 3.5.2. *The family $\mathcal{F} \to B$ is a marked (n-punctured) holomorphic family of Riemann surfaces*

Proof. This is a bit brief but missing details appear in other places.

The idea is to map $\mathcal{F}$ into the universal family $V(G)$.

Let $\sigma_t = f^{\mu_t} \circ (g_t)^{-1} : \Sigma_t \to \Sigma^{\mu_t}$ and define

$$\begin{aligned} (h, F) : \mathcal{F} &\longrightarrow V(G) \\ (t, z) &\longmapsto ([\mu_t], \sigma_t(z)) \end{aligned}$$

Note that h is the classifying map as in Section 2.5.3. Also $h : B \to T(G)$ is the mapping between the base spaces.

In our case h is holomorphic because by assumption $t \mapsto \mu_t$ is holomorphic and $\mu_t \mapsto [\mu_t]$ is holomorphic because it is the fundamental projection.

For t fixed, σ_t is a biholomorphism. Using this we see that (Σ, g_t, Σ_t) is Teichmüller equivalent to $(\Sigma, f^{\mu_t}, \Sigma^{\mu_t})$. This means that (h, F) is marking preserving.

Pulling back the complex structure by (h, F) makes $\mathcal{F}$ into a complex manifold. The complex structure of the fibers has not been changed as fiberwise F is the biholomorphism σ_t.

Therefore $\mathcal{F} \to B$ is a marked holomorphic family of Riemann surfaces.

□

Remark 3.5.3. There should be a more direct way to show that $\mathcal{F}$ is a marked holomorphic family. In that case F is the unique morphism of marked families guaranteed by the universality of $V(G)$. See Theorem 2.5.6.

Chapter 4

Holomorphicity of the Sewing Operation

In conformal field theory sewing is the fundamental operation. Understanding this operation and proving its holomorphicity is therefore crucial to further developments. The sewing isomorphism of determinant lines induced from the geometric sewing operation is addressed in Chapter 5. To prove that the isomorphism is holomorphic we also need a good understanding of the sewing operation on the geometric level.

Since the complex structure of Teichmüller space is understood through quasiconformal mappings, it is natural to try to formulate sewing in these term. Further motivation for this approach comes from Schiffer variation which is actually a special kind of sewing operation. Actually, the methods used in this chapter were inspired by the interpretation of Schiffer variation as a quasiconformal deformation of complex structure (see Gardiner [20] and Nag [44]).

The important preliminary results of this chapter are Theorem 4.5.2, Corollaries 4.5.3 and 4.5.4, and Proposition 4.5.7. The main result, that the sewing operation gives a holomorphic operation between Teichmüller spaces, appears in Theorem 4.6.2.

4.1 Riemann surfaces with parametrized boundaries and the sewing operation

In this section we closely follow Huang [30, D.3].

Let Σ be a compact Riemann surface with boundary and note that each boundary component is a one-dimensional real compact manifold. The orientation of the surface can be used to induce an orientation of each boundary component. A specified orientation of a boundary component is called positive (negative) if it agrees (disagrees) with the orientation induced from the surface.

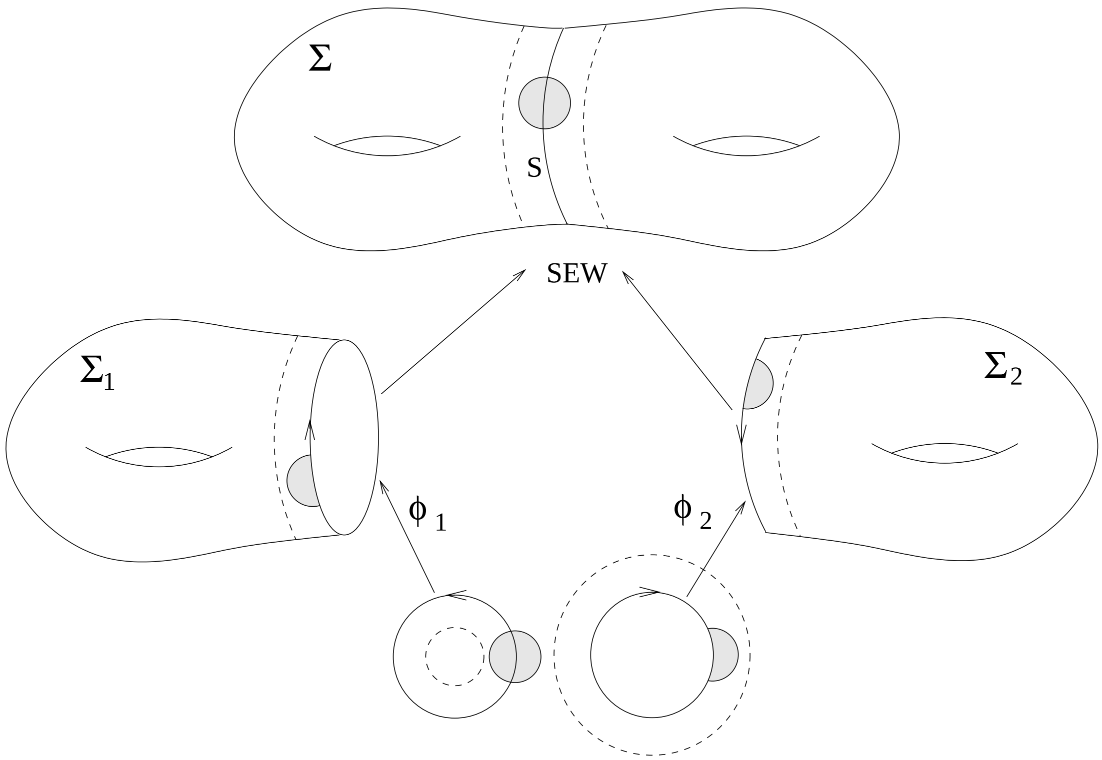


Figure 4.1: The sewing operation

Let γ be a boundary component of Σ. A map $\phi : S^1 \to \gamma$ called an *analytic parametrization* of the boundary component if ϕ can be extended to an analytic map (also called ϕ) from an annulus to Σ. Let $\mathbb{A}_{r_1}^{r_2} = \overline{B(0,r_2)} \setminus B(0,r_1) \subset \mathbb{C}$, for $r_1 < r_2$, be the annulus with inner radius r_1 and outer radius r_2. There are two possibilities for the extension of ϕ. Either the annulus is $\mathbb{A}_r^1$ where $r < 1$ or the annulus $\mathbb{A}_1^r$ where $r > 1$.

From the standard orientation of S^1, the analytic parametrization ϕ prescribes an orientation of the boundary component. We note that if $r < 1$ then the orientation is positive and if $r > 1$ then the orientation is negative.

We now describe the sewing operation as illustrated in Figure 4.1 . See for example Ahlfors and Sario [2] for a more detailed description. Let Σ_1 and Σ_2 be Riemann surfaces with analytically parametrized boundaries. Let γ_1 and γ_2 be boundary components of Σ_1 and Σ_2 respectively with analytic parametrizations ϕ_1 and ϕ_2. Assume that γ_1 is positively oriented and γ_2 is negatively oriented. The boundaries γ_1 and γ_2 can be identified by $(\phi_2)^{-1} \circ \phi_1 : \gamma_1 \to \gamma_2$. Note that this map is orientation reversing as is required in general (see [2]). In Figure 4.1 the arrows on the boundaries always indicate the orientation induced from the surface. We now define

$$\Sigma_1 \# \Sigma_2 = \Sigma_1 \sqcup \Sigma_2 / \sim$$

where $\sqcup$ stands for disjoint union and $x \in \gamma_1 \sim y \in \gamma_2$ if and only if $y = (\phi_2)^{-1} \circ \phi_1(x)$. Note that $\Sigma_1 \# \Sigma_2$ depends on the parametrizations. It is precisely this dependence that we wish to understand.

The local coordinate charts on $\Sigma_1 \# \Sigma_2$ only vary in a neighborhood of the sewn boundaries. Let S be the curve on $\Sigma_1 \# \Sigma_2$ corresponding to the sewn boundaries (see Figure 4.1). Let W be an open neighborhood of a point $z \in S^1$. Let W_1 be the part of W that is inside S^1, and W_2 the part outside. Then $\phi(W_1) \cup \phi(W_2) \subset \Sigma_1 \# \Sigma_2$ is by definition a chart on $\Sigma_1 \# \Sigma_2$.

4.2 Local coordinates and boundary parametrizations

Let Σ be a Riemann surface of conformal type (g, n). Recall from Section 3.1 that a local coordinate is a holomorphic map from a neighborhood of the puncture to a neighborhood of either 0 or ∞, depending on the orientation of the puncture. The space of local coordinates on Σ is $\mathcal{O}(\Sigma)$.

Let $\mathcal{O}_B(\Sigma) \subset \mathcal{O}(\Sigma)$ be the space of local coordinates with the following properties.

1. The image of each local coordinate contains the punctured unit disk.
2. The domains of the local coordinates do not intersect.

Depending on the orientation of the local coordinate, the punctured unit disk will be centered at 0 or ∞ in $\hat{\mathbb{C}}$. With an abuse of notation we use D to stand for either of these punctured open disks.

If $\phi = (\phi_1, \ldots, \phi_n) \in \mathcal{O}_B(\Sigma)$, then

$$\Sigma_B = \Sigma \setminus \left(\bigcup_{i=1\ldots n} (\phi_i)^{-1}(D) \right)$$

is a Riemann surface with analytically parametrized boundary components. The maps $(\phi_i)^{-1} : S^1 \to \partial \Sigma_B$ are the parametrizations. Note that the "B" in $\mathcal{O}_B(\Sigma)$ stands for “boundary”. All Riemann surfaces with analytically parametrized boundaries can be constructed in this way. To see this it is enough to note that parametrizations of boundary components can be used to sew in punctured unit disks to produce a surface with punctures and local coordinates.

Throughout this chapter, the term *local coordinate* will mean an element of the reduced space $\mathcal{O}_B(\Sigma) \subset \mathcal{O}(\Sigma)$. Let

$$\widetilde{T}_B(\Sigma) = \{[\Sigma, f, \Sigma, \phi] \in \widetilde{T}_B(\Sigma) \,|\, \phi \in \mathcal{O}_B(\Sigma)\},$$

and note it is a complex submanifold of $\widetilde{T}_B(\Sigma)$.

Remark 4.2.1. In this chapter we could consider local coordinates with arbitrary images. The only restriction needed is that the domains do not overlap. However, when the determinant line bundle is considered in Chapter 5 we must restrict to $\mathcal{O}_B(\Sigma)$.

4.3 Deformations in the plane

Recall that the basic idea behind Schiffer variation is to show that the varied surface is related to the original surface by a quasiconformal map and that this map depends analytically on the deformation parameter. In Schiffer variation the deformed disk is sewn into the surface via a very special boundary parametrization, namely $z \mapsto z + \epsilon z^{-1}$. In the general sewing of Riemann surfaces with analytically parametrized boundaries the variation occurs on an *annulus*. We therefore need to produce quasiconformal maps between annuli with different shaped boundary. This map will then used to produce a quasiconformal maps between differently sewn surfaces in Section 4.5.

We proceed by first constructing the map between annuli in the complex plane as illustrated in Figure 4.2. Recall that $\mathbb{A}_{r_1}^{r_2}$ is the annulus in the complex plane with inner radius r_1 and outer radius r_2. Let $\phi : \partial B(0, r_2) \to \mathbb{C}$ be the restriction of a biholomorphic map with the additional property that $|\phi(z)| > r_1$. Let $\mathbb{A}_{r_1}^{\phi}$ be the annular region between the curves $\partial B(0, r_1)$ and $\phi(\partial B(0, r_2))$ (see Figure 4.2).

Given an analytic family, $\phi_t(z)$, of such biholomorphic maps, our goal is to construct an analytic family of quasiconformal maps

$$H_t : \mathbb{A}_{r_1}^{r_2} \to \mathbb{A}_{r_1}^{\phi_t}$$

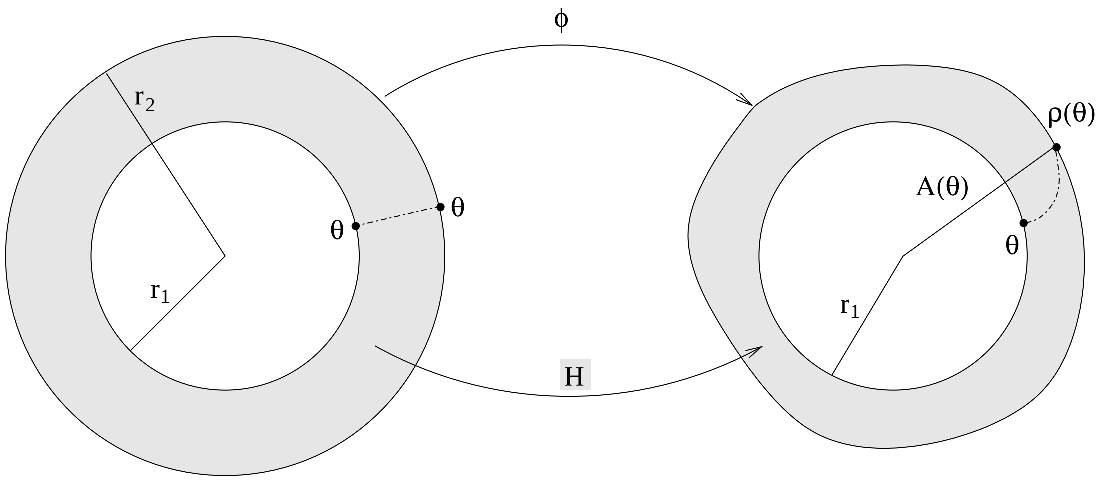


Figure 4.2: Annulus deformation

such that H_t is the identity on the inner circle and is equal to ϕ_t on the outer circle.

Before we start this construction we need to make precise the notion of an analytic family of maps, $\phi_t(z)$, and discuss some of their properties.

4.3.1 Holomorphic deformations of S^1

Lemma 4.3.1. *Any orientation preserving biholomorphic map $\phi : \partial B(0, r_2) \to \mathbb{C}$ such that $|\phi(z)| > r_1$ can be written as*

$$\phi(r_2 e^{i\theta}) = A(\theta) e^{i\rho(\theta)}$$

where $A(\theta)$ and $\rho(\theta)$ are smooth functions. Moreover, $\rho(\theta+2\pi) = \rho(\theta)+1$ and $\rho'(\theta) > 0$.

Proof. For notational convenience, let $z = r_2 \mathrm{e}^{i\theta}$. From general theory of loop groups ([46], page 59)

$$\frac{\phi(z)}{|\phi(z)|} = \mathrm{e}^{i\rho(\theta)}$$

where ρ is a smooth function and $\rho(\theta + 2\pi) = \rho(\theta) + 1$. The '1' is the winding number of ϕ. Because ϕ is orientation preserving and injective $\rho'(\theta) > 0$.

Note that $|\phi(z)|$ is a smooth function and is bounded away from zero, as by assumption $|\phi(z)| > r_1$. So $A(\theta) = |\phi(z)|^{1/2}$ is also a smooth function as required. □

As we are interested in the analyticity of the sewing operation, we are interested in those ϕ which are holomorphic perturbations of the identity. To make this precise we need to recall the notion of a *holomorphic motion* (see Definition 2.2.1 on page 12). In our case we have much stronger conditions, as each member of the family will be biholomorphic (not just injective as in the definition). The following lemma just collects some simple facts that will be needed later.

Lemma 4.3.2. *Let $\phi_t(z)$ be a holomorphic motion of $\partial B(0, r_2)$ such that for each fixed t, $\phi_t(z)$ is holomorphic as a function of z. Then $A_t(\theta)$ and $\rho_t(\theta)$, defined by $\rho_t(\theta) = A_t(\theta)\exp(i\rho_t(\theta))$, are analytic in t for each fixed θ and are smooth as functions of (t, θ). Moreover,*

$$\frac{\partial A_t}{\partial \theta} \quad \textit{and} \quad \frac{\partial \rho_t}{\partial \theta}$$

are holomorphic functions of t for each fixed θ.

Proof. By assumption, $\phi_t(z)$ is holomorphic in t and z separately and so by Hartogs' theorem (Theorem 2.2.4) it is a holomorphic function of two variables. This immediately implies that $A_t(\theta)$ and $\rho_t(\theta)$ are smooth as functions of two variables.

It remains to show the analyticity in t. For each fixed z, direct computation gives

$$\frac{\partial \phi_t(z)}{\partial \bar{t}} = \left[\frac{\partial A_t(\theta)}{\partial \bar{t}} + iA_t(\theta)\frac{\partial \rho_t(\theta)}{\partial \bar{t}}\right] \mathrm{e}^{i\rho_t(\theta)} = 0$$

Equating the real and imaginary parts to zero, and noting that $A_t(\theta) \neq 0$, gives

$$\frac{\partial A_t(\theta)}{\partial \bar{t}} = 0 \quad \text{and} \quad \frac{\partial \rho_t(\theta)}{\partial \bar{t}} = 0$$

which implies analyticity in t.

Proposition 2.2.1 now applies and so $A_t'(\theta)$ and$\rho_t'(\theta)$ are analytic in t. □

4.3.2 Annulus Deformation

Let $\phi_t(z)$ be an analytically family satisfying the conditions in Lemma 4.3.2 and such that $|\phi_t(z)| > r_1$. We want to produce a quasiconformal map

$$H_t : \mathbb{A}_{r_1}^{r_2} \to \mathbb{A}_{r_1}^{\phi_t}$$

depending holomorphically on t, and with boundary values specified by

$$H_t(z) = \begin{cases} z & , \quad z = r_1\mathrm{e}^{i\theta} \\ \phi_t(z) & , \quad z = r_2\mathrm{e}^{i\theta}. \end{cases}$$

That is, H_t is the identity on the inner circle and is equal to ϕ_t on the outer circle.

The holomorphic dependence on t is needed because $H_t(z)$ will be used to construct a map between surfaces produced by sewing with ϕ_0 and ϕ_t.

Remark 4.3.3. For general ϕ, such an H may not be a homeomorphism and thus certainly not quasiconformal. We will see in two different ways why H_t must be quasiconformal for sufficiently small t. One proof is elementary and relies on the fact that ϕ_t is close to the identity for small t. The second proof is as a corollary of the λ-lemma of Mañé, Sad and Sullivan [41] (see Theorem 2.2.6). It is interesting to note that the conditions that need to be imposed on the family ϕ_t will be identical in each case. This will be discussed further.

We now define H_t is the simplest possible way using linear deformation in the rotational and radial directions. This is done by writing

$$H_t(re^{i\theta}) = R_t(r,\theta)e^{i\lambda_t(r,\theta)}$$

for $r_1 \leq r \leq r_2$.

Remark 4.3.4. The best way to visualize H is to represent the annulus $\mathbb{A}^{\phi}_{r_1}$ in the (r,θ)-plane. See Figure 4.3. The straight lines in this figure are the images of the radial line segments joining $(r1,\theta')$ to (r_2,θ') in the complex plane (see Figure 4.2). The curve is the image of $\partial B(0,r_2)$ and is described in parametric form by $(A(\theta),\rho(\theta))$. Recall that by assumption $\rho'(\theta) > 0$ so the curve is strictly increasing.

We define R and λ as follows. The rotational twist is defined by

$$\lambda_t(r,\theta) = \frac{\rho_t(\theta)(r-r_1) + (r_2-r)\theta}{r_2-r_1} \tag{4.1}$$

for $r_1 \leq r \leq r_2$. For each fixed θ, this is the formula in the (r,λ)-plane for the line segment joining (r_1,θ) and $(r_2,\rho(\theta))$. Note that, $\lambda_0(r,\theta) = \theta$.

The deformation in the radial direction is defined by

$$R_t(r,\theta) = \frac{1}{r_2-r_1}\left[(r_2-r)r_1 + (r-r_1)A_t(\theta)\right] \tag{4.2}$$

For each fixed θ this is just a linear stretching of the radial segment $\overline{r_1r_2}$ to $\overline{r_1A(\theta)}$. It is easy to check that $R_0(r,\theta) = r$.

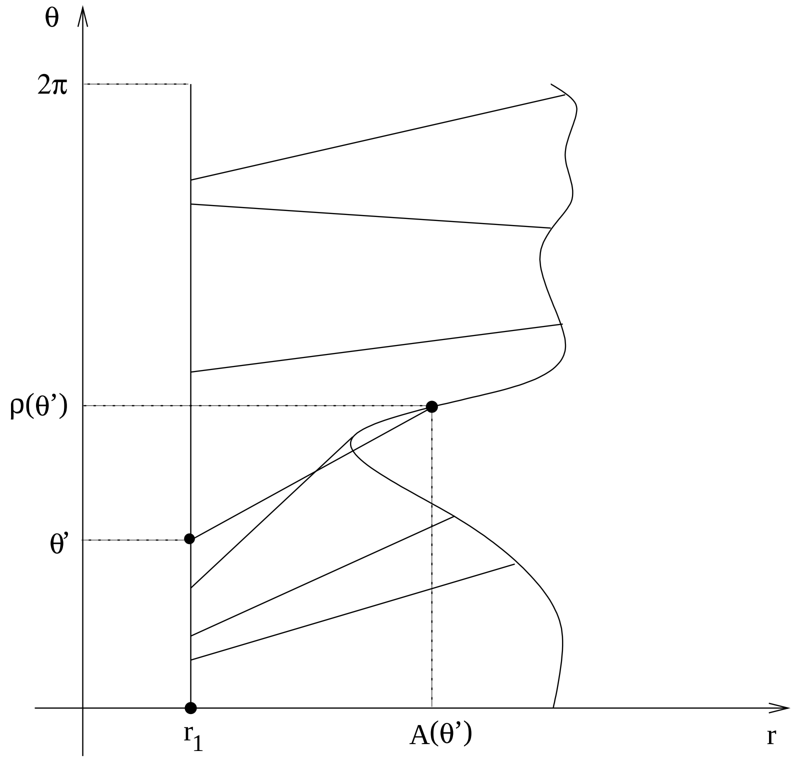


Figure 4.3: Deformed annulus as a strip

Now let

$$H_t(re^{i\theta}) = R_t(r,\theta)e^{i\lambda_t(r,\theta)} \tag{4.3}$$

and observe that the boundary conditions are satisfied. Note that H_0 is the identity.

To show that $H_t(z)$ is quasiconformal, either directly or by the λ-lemma, we first have to show that $H_t(z)$ is injective and in fact has the correct image. This is actually the main non-trivial part and so we tackle it first.

From Figure 4.3, it is easy to see that $\mathbb{A}_{r_1}^{\phi}$ is contained in the image of H. However if the curve is varying too much we see that the image (part of a line segment) may lie outside the strip. In this case we also see that H is not injective (two lines cross). Actually it is not hard to see that if the image does not fall outside the strip then H will automatically be a bijection.

The geometric condition imposed on the curve is that the slope of any line segment must be less that the slope of the tangent line. Algebraically this translates into the inequality

$$\left|\frac{\rho(\theta)-\theta}{A(\theta)-r_1}\right| < \left|\frac{\rho'(\theta)}{A'(\theta)}\right|.$$

Since $A(\theta) > r_1$ and $\rho'(\theta) > 0$, we can write this as $|A'(\theta)|\,|\rho(\theta)-\theta| < (A(\theta)-r_1)\,\rho'(\theta)$. Actually we can see from Figure 4.3 that the only case we need to consider is when either $A'(\theta)$ and $(\rho(\theta)-\theta)$ are both positive or both negative. Hence we can drop the absolute value signs and write

$$A'(\theta)\,(\rho(\theta)-\theta) < (A(\theta)-r_1)\,\rho'(\theta). \tag{4.4}$$

Informally we see that for ϕ *close to the identity*, $A(\theta) \approx r_2$ and $\rho(\theta) \approx \theta$, so we should have $A' \approx 0$, $\rho(\theta)-\theta \approx 0$, $A(\theta)-r_1 \approx r_2-r_1$, and $\rho'(\theta) \approx 1$. If this is the case then the inequality holds.

We now make this precise. The appropriate estimates rely on the joint smoothness of $\phi_t(z)$ in t and z.

Lemma 4.3.5. *Let $\phi_t(z)$ be a holomorphic motion of $\partial B(0,r_2)$ such that for each fixed t, $\phi_t(z)$ is orientation preserving and biholomorphic as a function of z. Then $H_t(z) : \mathbb{A}_{r_1}^{r_2} \to \mathbb{A}_{r_1}^{\phi_t}$ is a diffeomorphism for sufficiently small t.*

Proof. First we note that $R_t(r,\theta)$ and $\lambda_t(r,\theta)$, as defined in equations (4.1) and (4.2), are smooth functions of t, r and θ.

For H_t to be defined we need $|\phi_t(z)| > r_1$. Since $|\phi_t(z)| = A(\theta)$ and we will be bounding $|A(\theta)-r_2|$ from above, there is no need to provide a separate argument.

By the above discussion $H_t(z)$ is a homeomorphism if inequality (4.4) holds. So we need to produce bounds on the quantities $A_t(\theta), A_t'(\theta), \rho_t(\theta)$and $\rho_t'(\theta)$ that are independent of θ. This is where the joint smoothness is needed. The required bounds follow directly by repeated application of Proposition 2.2.2.

For example we know that $A_t(\theta)$ is smooth and $A_0(\theta) = r_2$. Therefore Proposition 2.2.2 applies to $A_t(\theta)-r_2$ and we conclude that $A_t(\theta)-r_2$ can be made arbitrarily close to zero by taking $|t|$ sufficiently small. This makes precise the above informal statement $A(\theta) \approx r_2$.

Similarly we can make the other informal statements precise. Thus inequality 4.4 holds for $|t|$ sufficiently small and $H_t(z)$ is a homeomorphism.

The smoothness of $H_t(z)$ follows directly from the smoothness of $R_t(r,\theta)$ and $\lambda_t(r,\theta)$. So $H_t(z)$ is a diffeomorphism as required. □

Proposition 4.3.6. *For each fixed z the dependence of $H_t(z)$ on t is analytic.*

Proof. This follows by direct calculation.

$$\begin{aligned}\frac{\partial H_t(z)}{\partial \bar{t}} &= \left[\frac{\partial R_t(z)}{\partial \bar{t}} + iR_t(z)\frac{\partial \rho_t(z)}{\partial \bar{t}}\right] \mathrm{e}^{i\lambda_t} \\ &= \left[\frac{r-r_1}{r_2-r_1}\frac{\partial A_t(z)}{\partial \bar{t}} + iR_t(z)\frac{r-r_1}{r_2-r_1}\frac{\partial \rho_t(z)}{\partial \bar{t}}\right] \\ &= 0+0\end{aligned}$$

where the last equality follows from Lemma 4.3.2. □

Proposition 4.3.7. *The map $H_t(z)$ defined by Equation 4.3 is quasiconformal in z for each fixed t.*

Proof. (method 1) By the previous lemmas we have shown that H is holomorphic motion of $\partial B(0,r_2)$ and thus by the λ-lemma (Theorem 2.2.6), H is quasiconformal. □

This is very nice, but H can be shown to be quasiconformal by directly computing its complex dilation. This completely elementary proof will be given as it illustrates an interesting point which we will note after the proof.

Proof. (method 2)

Since we already know that H is an orientation preserving diffeomorphism it is sufficient to show its the complex dilation, $\mu_H(z)$, is bounded by 1. That is, $||\mu_H(z)||_\infty < 1$. This will be done by explicitly computing the complex dilation using the chain rule.

Let $z = r\mathrm{e}^{i\theta}$. Then

$$\frac{\partial r}{\partial z} = \frac{1}{2}\mathrm{e}^{-i\theta} \quad , \quad \frac{\partial r}{\partial \bar{z}} = \frac{1}{2}\mathrm{e}^{i\theta} \quad , \quad \frac{\partial \theta}{\partial z} = \frac{-i}{2r}\mathrm{e}^{-i\theta} \quad \text{and} \quad \frac{\partial \theta}{\partial \bar{z}} = \frac{i}{2r}\mathrm{e}^{i\theta}$$

and so

$$\begin{aligned}\frac{\partial H}{\partial z} &= \frac{\partial H}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial H}{\partial \theta}\frac{\partial \theta}{\partial z} \\ &= \left(\frac{\partial R}{\partial r}\mathrm{e}^{i\lambda} + iR\frac{\partial \lambda}{\partial r}\mathrm{e}^{i\lambda}\right)\frac{1}{2}\mathrm{e}^{-i\theta} + \left(\frac{\partial R}{\partial \theta}\mathrm{e}^{i\lambda} + iR\frac{\partial \lambda}{\partial \theta}\mathrm{e}^{i\lambda}\right)\frac{i}{2r}\mathrm{e}^{-i\theta} \\ &= \frac{1}{2}\mathrm{e}^{i(\lambda-\theta)}\left[\left(\frac{\partial R}{\partial r} + \frac{R}{r}\frac{\partial \lambda}{\partial \theta}\right) + i\left(R\frac{\partial \lambda}{\partial r} - \frac{1}{r}\frac{\partial R}{\partial \theta}\right)\right].\end{aligned}$$

Similarly

$$\frac{\partial H}{\partial \bar{z}} = \frac{1}{2}\mathrm{e}^{i(\lambda-\theta)}\left[\left(\frac{\partial R}{\partial r} - \frac{R}{r}\frac{\partial \lambda}{\partial \theta}\right) + i\left(R\frac{\partial \lambda}{\partial r} + \frac{1}{r}\frac{\partial R}{\partial \theta}\right)\right]$$

A short calculation gives

$$\frac{\partial R}{\partial r} = \frac{1}{r_2 - r_1}(A(\theta) - r_1) \quad \text{and} \quad \frac{\partial R}{\partial \theta} = \frac{r - r_1}{r_2 - r_1}\frac{dA}{d\theta}.$$

Note that

$$\mu_H(z) = \frac{\partial H}{\partial \bar{z}} / \frac{\partial H}{\partial z}$$

is of the form

$$\frac{(A+B) + i(C-D)}{(A-B) + i(C+D)},$$

and that

$$\begin{aligned}\left|\frac{(A+B) + i(C-D)}{(A-B) + i(C+D)}\right|^2 < 1 &\Longleftrightarrow (A-B)^2 + (C+D)^2 < (A+B)^2 + (C-D)^2 \\ &\Longleftrightarrow CD < AB\end{aligned}$$

We now see that

$$\begin{aligned}|\mu_F(z)| < 1 &\Longleftrightarrow R\frac{\partial \lambda}{\partial r}\frac{1}{r}\frac{\partial R}{\partial \theta} < \frac{\partial R}{\partial r}\frac{R}{r}\frac{\partial \lambda}{\partial \theta} \quad , \quad \forall \theta \in [0, 2\pi] \\ &\Longleftrightarrow (\rho(\theta) - \theta)\frac{dA}{d\theta} < (A(\theta) - r_1)\frac{d\rho}{d\theta} \quad , \quad \forall \theta \in [0, 2\pi]\end{aligned}$$

This last inequality is true as it is identical to inequality (4.4) which we have already shown is true for sufficiently small t. Hence $H_t(z)$ is a quasiconformal homeomorphism for sufficiently small t as desired..

□

Remark 4.3.8. The fact that two very different calculations both led to inequality (4.4) is interesting. Examining Figure 4.3 more carefully we see that if H is not injective then two lines cross. So as a map in the $(r, \theta)-plane$, H maps a rectangle to a region between the two crossed line. As the intersection point is approached the circular dilation will go to infinity. In fact it is not too hard to see that these conditions are equivalent.

As $H_t(z)$ will be used to prove holomorphicity of the sewing we will eventually need the fact that the complex dilation μ_{H_t} depends analytically on t. This is straightforward.

Corollary 4.3.9. *The complex dilation μ_{H_t} of $H_t(z)$ is analytic in t.*

Proof. $H_t(z)$ is a diffeomorphism by Lemma 4.3.5 and is analytic in t by Proposition 4.3.6. Therefore Proposition 2.2.1 applies and so $\partial H/\partial \bar{z}$ and $\partial H/\partial z$ are analytic in t.

We can come to the same conclusion more directly. The chain rule calculations in the proof of Proposition 4.3.7 show that

$$\frac{\partial H_t}{\partial z} \quad \text{and} \quad \frac{\partial H_t}{\partial \bar{z}}$$

only depend on t through $A_t(\theta)$, $A_t'(\theta)$, $\rho_t(\theta)$ and $\rho_t'(\theta)$. By Lemma 4.3.2 all these quantities are holomorphic in t. Therefore

$$\mu_{H_t} = \frac{\partial H_t}{\partial \bar{z}} / \frac{\partial H_t}{\partial z}$$

is holomorphic in t □

4.4 Lifting to the surface

We want to use the quasiconformal map H between annuli to produce a quasiconformal map between Riemann surfaces.

A family of surfaces is produced in the following way (see Figures 4.4 and 4.5). Assume for simplicity that Σ has a single puncture p. Let $\phi_t \in \mathcal{O}_B(\Sigma)$ (see Section 4.2) be an analytic family of local coordinates at a puncture $p \in \Sigma$. We further assume that p is negatively oriented so that ϕ maps to a neighborhood of $0 \in \mathbb{C}$. Let $\Delta_0 = B(0,1)\backslash\{0\}$ be the punctured unit disk centered at 0.

Let $B_t = \phi_t^{-1}(\Delta_0)$, and let $\gamma_t = \phi_t^{-1}(S^1)$. Let

$$\Sigma_t = \Sigma \setminus B_t$$

with boundary parametrization given by $(\phi_t)^{-1}$.

Remark 4.4.1. The choice of orientation of p was arbitrary. What follows works equally well for a positively oriented boundary parametrization.

Remark 4.4.2. In Figure 4.4 the curves γ_0 and γ_t intersect as is generally the case. For clarity of Figure 4.5, the curve γ_t has been drawn disjoint from γ_0. Mathematically we impose no such restriction.

We introduce some convenient notation. If a and b are two simple closed curves on Σ that bound an annular region then we denote that region by $\mathbb{A}_a^b$. This is a generalization of the notation used for annuli in the plane.

Consider simple closed analytic curves c and c' about $p \in \Sigma$ such that c' is *inside* c and $\gamma_t \subset \mathbb{A}_{c'}^c$ for all t. See Figure 4.4. Let $A_t = \mathbb{A}_{\gamma_t}^c$ be the annular region on Σ bounded by c and γ_t. Let $G : \mathbb{A}_{c'}^c \to \mathbb{C}$ be a biholomorphic map such that $G(c) = \partial B(0, r_1)$ and $G(\gamma_0) = \partial B(0, r_2)$ for some $r_1 < r_2$ (see Figure 4.5). These conditions are equivalent to requiring $G(A_0) = \mathbb{A}_{r_1}^{r_2}$. Such a map always exists by the genus-zero uniformization theorem (otherwise known as the Riemann mapping theorem). Note that is no condition imposed on $G(c')$.

Note that $\phi_t^{-1} \circ \phi_0$ is an orientation preserving map from $\gamma_0 \subset \partial\Sigma_0$ to $\gamma_t \subset \partial\Sigma_t$. Let $\beta_t : G(\gamma_0) = \partial B(0, r_2) \to G(\gamma_t)$ be defined by

$$\beta_t(z) = \left(G \circ \phi_t^{-1} \circ \phi_0 \circ G^{-1}\right)(z).$$

It follows that β is orientation preserving and a holomorphic function of both t and z because it is the composition of such maps. In particular we note that $\beta_t(z)$ is a holomorphic motion of $\partial B(0, r_2)$. By Proposition 4.3.7 there exists a quasiconformal map

$$H_t(z) : \mathbb{A}_{r_1}^{r_2} = G(A_0) \to G(A_t)$$

which is analytic in t and satisfies the boundary conditions, $H_t(z) = z$ for $z = r_1 \exp(i\theta)$ and $H_t(z) = \beta_t(z)$ for $z = r_2 \exp(i\theta)$.

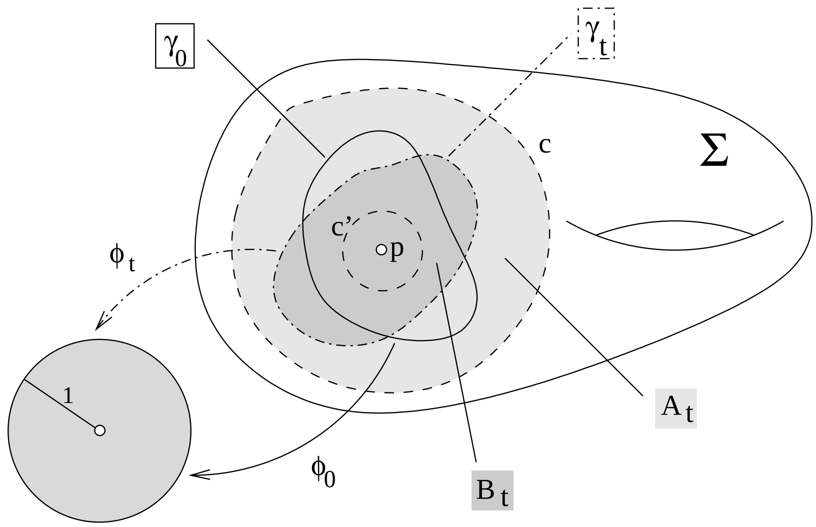


Figure 4.4: Family of local coordinates

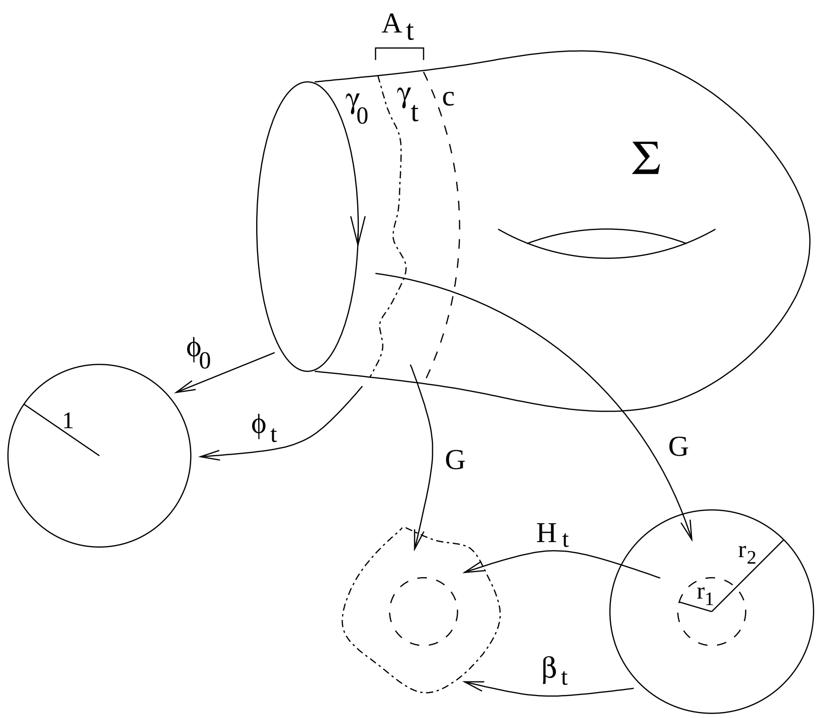


Figure 4.5: Deforming the boundary

Proposition 4.4.3. *The map $\Phi_t : \Sigma_0 \to \Sigma_t$ defined by*

$$\Phi_t = \begin{cases} \textit{Identity} & \textit{on} \quad \Sigma \setminus (A_0) \\ G^{-1} \circ H_t \circ G & \textit{on} \quad A_0 \end{cases}$$

is quasiconformal and depends holomorphically on t.

Remark 4.4.4. As Σ_0 and Σ_t are subsets of Σ is makes sense to talk about the *identity* map as well as holomorphicity in t.

Proof. We first show that Φ_t is well defined. For $x \in c$, $G(x) \in \partial B(0, r_1)$, and since H_t is the identity on $\partial B(0, r_1)$ we have $(G^{-1} \circ H_t \circ G)(x) = G^{-1}(G(x)) = x$. Because $H_t(z)$ is analytic in t for each fixed z, and the other maps are independent of t, we see that $\Phi_t(z)$ is analytic in t.

The map $\Phi_t(z)$ is quasiconformal because it is defined by a composition of conformal and quasiconformal maps. □

Corollary 4.4.5. *The complex dilation μ_{Φ_t} of Φ_t is holomorphic in t.*

Proof. The proof essential the same as the proof of Corollary 4.3.9. Observe that in local coordinates $G^{-1} \circ H_t \circ G$ is smooth as a function of z and t and is analytic in t because $H_t(z)$ is. Therefore, by either a chain rule calculation, or Proposition 2.2.1, the derivatives of $G^{-1} \circ H_t \circ G$ with respect to z and $\bar{z}$ are holomorphic in t. Thus

$$\mu_{\Phi_t} = \frac{\partial \Phi_t}{\partial \bar{z}} / \frac{\partial \Phi_t}{\partial z}$$

is holomorphic in t as desired.

□

4.5 Sewing

Consider two Riemann surfaces Σ^1 and Σ^2 with varying boundary parametrizations as in Section 4.4. If the parametrizations are oppositely oriented then these surfaces are compatible for sewing. We want to show that the resultant surface depends analytically on the parametrizations. Figure 4.6 should help in understanding the notation and constructions. As in Figure 4.5 the curves γ_t have been drawn disjoint from γ_0. This is only to make the picture easier to draw and is not a mathematical requirement. Note that the arrows on the boundary components show the orientation induced from the surface.

For simplicity we assume that Σ^1 and Σ^2 have oppositely-oriented punctures p_1 and p_2 respectively, and no others. Let $\phi^1_{t_1} \in \mathcal{O}_B(\Sigma_1)$ and $\phi^2_{t_2} \in \mathcal{O}_B(\Sigma_1)$ be analytic families of local coordinates at the punctures p_1 and p_2 respectively.

Recall that D stands for the punctured unit disk centered at 0 or ∞. For $i = 1, 2$, let $B^i_{t_i} = (\phi^i_{t_i})^{-1}(D)$ and let $\gamma^i_{t_i} = (\phi^i_{t_i})^{-1}(S^1)$. Consider the Riemann surfaces

$$\Sigma^i_{t_i} = \Sigma^i \setminus B^i_{t_i}$$

with boundary parametrizations given by $\phi^i_{t_i}$. Because of the opposite orientation of the parametrizations these boundaries are compatible for sewing.

Let $\Sigma_{t_1,t_2} = \Sigma^1_{t_1} \# \Sigma^2_{t_2}$. Our goal is to produce a quasiconformal map

$$\Phi_{t_1,t_2} : \Sigma_{0,0} \longrightarrow \Sigma_{t_1,t_2}$$

that depends holomorphically on t_1 and t_2. What is meant here by "holomorphically" will be made precise.

Remark 4.5.1. The map Φ_{t_1,t_2} is analogous to the Schiffer variation map ν^ϵ. (See Section 2.6). The difference in our case is that the *variation* is occurring on an annular region on the Riemann surface rather than on a disk. Moreover the sewing is performed using arbitrary holomorphic families of parametrizations.

We will define c^i, A^i and G^i, for $i = 1, 2$, as in Section 4.4 (see also Figure 4.4). That is, for $i = 1, 2$, let c^i and c'^i be a simple closed analytic curves about $p^i \in \Sigma^i$ such that c'^i is *inside* c^i and $\gamma^i_{t_i} \subset \mathbb{A}^{c^i}_{c'^i}$ for all t_i. Let $A^i_t = \mathbb{A}^{c^i}_{\gamma^i_{t_i}}$ be the annular region on Σ^i bounded by c^i and $\gamma^i_{t_i}$. Let $G^i : \mathbb{A}^{c^i}_{c'^i} \to \mathbb{C}$ be biholomorphic maps such that $G^i(c^i) = \partial B(0, r^i_1)$ and $G(\gamma^i_0) = \partial B(0, r^i_2)$ for some $r^i_1 < r^i_2$.

By Section 4.3.2 there exist quasiconformal maps

$$H^i_{t_i} : \mathbb{A}^{r^i_2}_{r^i_1} \to \mathbb{A}^{\phi^i_{t_i}}_{r^i_1}$$

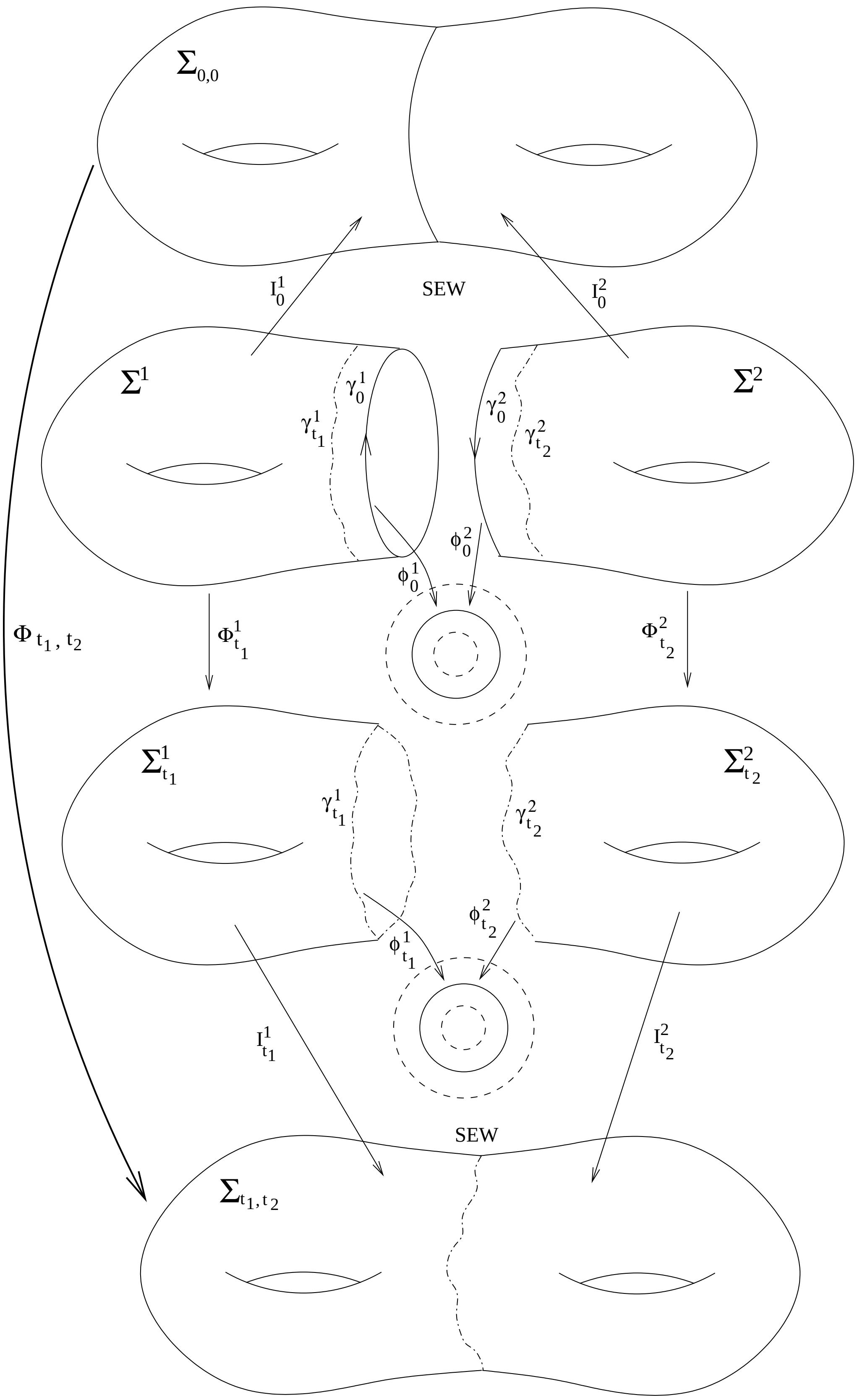


Figure 4.6: Varying the parametrization

depending holomorphically on t and with boundary values specified by

$$H^i_{t_i}(z) = \begin{cases} z & , \quad z = r^i_1 e^{i\theta} \\ \phi^i_{t_i}(z) & , \quad z = r^i_2 e^{i\theta}. \end{cases}$$

By Proposition 4.4.3 there exist quasiconformal maps

$$\Phi^i_{t_i} : \Sigma^i_0 \to \Sigma^i_{t_i}$$

that depend holomorphically on t_i.

Let I^i_0 be the inclusion maps of Σ^i_0 into $\Sigma_{0,0} = \Sigma^1_0 \# \Sigma^2_0$. Let $I^i_{t_i}$ be the inclusion maps of $\Sigma^i_{t_i}$ into $\Sigma_{t_1,t_2} = \Sigma^1_{t_1} \# \Sigma^2_{t_2}$.

Theorem 4.5.2. *The map*

$$\Phi_{t_1,t_2} = \begin{cases} I^1_{t_1} \circ \Phi^1_{t_1} \circ (I^1_0)^{-1} & \text{on} \quad I^1_0(\Sigma^1_0) \subset \Sigma_{0,0} \\ I^2_{t_2} \circ \Phi^2_{t_2} \circ (I^2_0)^{-1} & \text{on} \quad I^2_0(\Sigma^2_0) \subset \Sigma_{0,0} \end{cases} \tag{4.5}$$

$$= \begin{cases} I^i_{t_i} \circ (I^i_0)^{-1} = \text{“Identity”} & \text{on} \quad I^1_0(\Sigma^1 \setminus A^1_0) \cup I^2_0(\Sigma^2 \setminus A^2_0) \\ I^1_{t_1} \circ (G^1)^{-1} \circ H^1_{t_1} \circ G^1 \circ (I^1_0)^{-1} & \text{on} \quad I^1_0(A^1_0) \\ I^2_{t_2} \circ (G^2)^{-1} \circ H^2_{t_2} \circ G^2 \circ (I^2_0)^{-1} & \text{on} \quad I^2_0(A^2_0) \end{cases} \tag{4.6}$$

defines a quasiconformal homeomorphism from $\Sigma_{0,0}$ to Σ_{t_1,t_2}.

Proof. First we show that Φ_{t_1,t_2} is well defined. Choose $z \in S^1$ and let $x^i = (\phi^i_0)^{-1}(z) \in \gamma^i_0$. The sewing identifies x^1 and x^2 on $\Sigma_{0,0}$, so $I^1_0(x^1) = I^2_0(x^2)$.

The images of $I^1_0(x^1)$ and $I^2_0(x^2)$ under Φ_{t_1,t_2} are

$$\begin{aligned} I^1_0(x^1) \longmapsto \left(I^1_{t_1} \circ (G^1)^{-1} \circ H^1_{t_1} \circ G^1\right)(x) &= \left(I^1_{t_1} \circ (\phi^1_{t_1})^{-1} \circ \phi^1_0\right)(x^1) \\ &= \left(I^1_{t_1} \circ (\phi^1_{t_1})\right)(z) \end{aligned}$$

and

$$\begin{aligned} I^2_0(x^2) \longmapsto \left((G^2)^{-1} \circ H^2_{t_2} \circ G^2\right)(x) &= \left(I^2_{t_2} \circ (\phi^2_{t_2})^{-1} \circ \phi^2_0\right)(x^2) \\ &= \left(I^2_{t_2} \circ (\phi^2_{t_2})\right)(z). \end{aligned}$$

Now since the sewing of $\Sigma^1_{t_1}$ and $\Sigma^2_{t_2}$ is by the identification of $\phi^1_{t_1}(z)$ and $\phi^1_{t_1}(z)$ for $z \in S^1$ we see that these two images are equal on Σ_{t_1,t_2}.

We must also check that Φ_{t_1,t_2} is well defined on $I^i_0(c^i)$. This follows easily as H^i is the identity on $G^i(c^i)$.

It is now clear that Φ_{t_1,t_2} is a homeomorphism. It is quasiconformal because it is the composition of conformal and quasiconformal maps. □

Corollary 4.5.3. *The complex dilation $\mu(\Phi_{t_1,t_2})$ of Φ_{t_1,t_2} depends holomorphically on t_1 and t_2.*

Proof. This follows directly from Corollary 4.4.5 and the definition of Φ_{t_1,t_2}. □

The next corollary can be though of as the analogy of Theorem 2.6.3 from Schiffer variation that says $\epsilon \mapsto [\Sigma, \nu^\epsilon, \Sigma^\epsilon]$ is holomorphic.

Corollary 4.5.4. *The map*

$$\mathcal{L} : \mathbb{C} \times \mathbb{C} \longrightarrow T(\Sigma_{0,0})$$

given by

$$(t_1, t_2) \longmapsto [\Sigma_{0,0}, \Phi_{t_1,t_2}, \Sigma_{t_1,t_2}]$$

is holomorphic in t_i for $|t_i|$ sufficiently small, $i = 1, 2$.

Proof. From Theorem 2.3.4 it is enough to show that $\mu(\Phi_{t_1,t_2})$ is holomorphic in t_i, but this was just proved in Corollary 4.5.3. This proof is identical to the proof that the map $\epsilon \mapsto [\Sigma, \nu^\epsilon, \Sigma^\epsilon]$ in Schiffer variation is holomorphic. See Theorem 2.6.3 and the discussion preceding it. □

Conjecture 4.5.5. As in Schiffer variation one should be able to prove that the map $\mathcal{L}$ gives a holomorphic coordinate chart for $T(\Sigma_{0,0})$ if the ϕ^i depend on a total of $\dim(T(\Sigma_{0,0}))$ parameters.

Remark 4.5.6. The holomorphicity of $\mathcal{L}$ comes from the fact that the maps Φ_{t_1} and Φ_{t_2} are holomorphic on t_1 and t_2. In the genus-zero case in Huang [30, Section 1.4], the analogous maps are called $F^{(1)}$ and $F^{(2)}$ (see also Figure 3 which is analogous to our Figure 4.6). The main content of the higher-genus work is that Φ_{t_1,t_2} also carries information of variation in the *moduli direction.* In genus-zero the moduli direction is trivial and the maps F are conformal (as guaranteed by the Riemann mapping theorem).

If Σ_i is of conformal type (g_i, n_i) for $i = 1, 2$ then $\Sigma_{t_1}\#\Sigma_{t_2}$ is of type $(g_1 + g_2, n_1 + n_2 - 2)$. Recall that γ_0^i is the boundary of Σ_0^i to be sewn, and let S be the corresponding curve on $\Sigma_{0,0}$. Let N_1 and N_2 be the domains for the complex parameters t_1 and t_2 and let $N = N_1 \times N_2$. Recall that I_0^i are the inclusions of Σ_0^i into $\Sigma_{0,0}$. Let I_0 be the inclusion of $\Sigma_0^1 \sqcup \Sigma_0^2$ into $\Sigma_{0,0}$. Consider the family of Riemann surfaces

$$E = \coprod_{(t_1,t_2)\in N} \Sigma_{t_1}\#\Sigma_{t_2}.$$

From Section 2.5.3, recall the definition of an n-punctured holomorphic family and of a morphism between such families.

Proposition 4.5.7. *The family E over N together with the marking map $\Phi_{t_1,t_2} : \Sigma_{0,0} \to \Sigma_{t_1,t_2}$ forms an $(n_1 + n_2 - 2)$-punctured holomorphic family of Riemann surfaces.*

Proof. This could be proved with little effort using the construction in Section 3.5 and the fact that Φ_{t_1,t_2} depends holomorphically on t_1 and t_2. Instead we give a direct proof along the lines of the proof that Schiffer variation produces a holomorphic family.

We show that the family of surfaces E, is a complex manifold by producing coordinate charts. Away from an annular neighborhood of the curve S, the surfaces Σ_{t_1,t_2} are not changing (note that Φ_{t_1,t_2} is the identity away from S).

If (U_α, ζ_α) is a local coordinate chart on $\Sigma_{0,0}$ such that $U_\alpha \cap S = \emptyset$, then we can consider (U_α, ζ_α) as a coordinate chart on Σ_{t_1,t_2}. So we set $N \times U_\alpha$ with the map $(t_1, t_2, x) \mapsto (t_1, t_2, \zeta_\alpha(x))$ to be a local coordinate chart on the family E.

Let $z \in S^1$ and let $V \subset \mathbb{C}$ be a neighborhood of z. The circle S^1 divides V into two pieces, say V_1 and V_2. Then by definition of the sewing operation

$$U_{t_1,t_2} = (\phi_{t_1}^1)^{-1}(V_1) \sqcup (\phi_{t_2}^2)^{-1}(V_2)/\text{sewing identification}$$

gives a chart on Σ_{t_1,t_2}. We now let

$$\mathcal{U} = \coprod_{(t_1,t_2)\in N} U_{t_1,t_2}$$

with the map $(t_1, t_2, x) \mapsto (t_1, t_2, \phi^i_{t_i})$, where $i = 1$ or $i = 2$ depending on x, be a chart on E.

Since the $\phi^i_{t_i}$ depend holomorphically on t_i, all the transition functions will be holomorphic in t_i. □

Remark 4.5.8. The map $\mathcal{L} : N \to T(\Sigma_{0,0})$ in Corollary 4.5.4 can now be interpreted as the *classifying map* h. of Section 2.5.3.

Remark 4.5.9. We can think of Φ_{t_1,t_2} as a map from $\Sigma_{0,0}$ to E. As such, it now makes sense to say that Φ_{t_1,t_2} depends holomorphically on t_1 and t_2. This can be interpreted as a higher-genus analogue of Proposition 3.3.4 in Huang [30].

4.6 Sewing as a product on Teichmüller spaces

The main work of this chapter has already been done, but the sewing operation still needs to be formulated as an operation between Teichmüller spaces. Recall the definition of $\widetilde{T}_B(\Sigma)$ from Section 4.2. That is, we want a map $\widetilde{T}_B(\Sigma_1) \times \widetilde{T}_B(\Sigma_2) \longrightarrow \widetilde{T}_B(\Sigma_1 \# \Sigma_2)$. To define this map we must define the sewing of two Teichmüller space elements.

To avoid excessive subscripts we consider two Riemann surfaces X and Y and the corresponding rigged Teichmüller spaces $\widetilde{T}_B(X)$ and $\widetilde{T}_B(Y)$. Let $[X, f, X_1, \phi] \in \widetilde{T}_B(X)$ and $[Y, g, Y_1, \psi] \in \widetilde{T}_B(Y)$ where $\phi = (\phi_1, \ldots, \phi_m)$ and $\psi = (\psi_1, \ldots, \psi_n)$. Assume that ϕ_1 and ψ_1 have opposite orientation so that the surfaces can be sewn along the corresponding boundaries. We want to produce an element of the form $[X \# Y, l, X_1 \# Y_1, \phi']$ where $\phi' = (\phi_2, \ldots, \phi_m, \psi_2, \ldots, \psi_n)$ with some appropriate reordering.

To define $X \# Y$ we need local coordinates on X and Y. The definition of rigged Teichmüller space could be modified to include the extra data of local coordinates on the reference surface. Actually a choice of such local coordinates was already made when the complex structure was defined. In particular, these local coordinates are needed to give a trivialization of a neighborhood of the base point $[X, id, X]$.

Let $h = (h_1, \ldots, h_m)$ and $k = (k_1, \ldots, k_n)$ be a choice of local coordinates on the reference surfaces X and Y respectively. Then X and Y can be sewn using h_1 and k_1

to give $X\#Y$.

It remains to define the quasiconformal marking map $l : X\#Y \to X_1\#Y_1$. It is useful to look at Diagram 4.6. We would like $l|_X = f$ and $l|_Y = g$, but this does not give a well defined map because h_1 and k_1 are not related to ϕ_1 and ψ_1. Although we cannot construct a unique l we need the different choices to be homotopic, so that the point $[X\#Y, l, X_1\#Y_1, \phi'] \in \widetilde{T}_B(X\#Y)$ is well defined.

Let S be the curve on $X\#Y$ corresponding to the sewn boundaries of X and Y. Via the injection $I_X : X \to X\#Y$ we can consider $X \subset X\#Y$. Similarly we consider $Y \subset X\#Y$, $X_1 \subset X_1\#Y_1$ and $Y_1 \subset X_1\#Y_1$.

We claim that there exists maps $l : X\#Y \to X_1\#Y_1$ such that $l|_X$ is homotopic to f and $l|_Y$ is homotopic to g, and furthermore that

$$f^{-1} \circ l|_S : S \longrightarrow S$$

and

$$g^{-1} \circ l|_S : S \longrightarrow S$$

have winding number one. This last condition stops l having extra Dehn type twists in a neighborhood of S. Note that l depends on ϕ_1 and ψ_1 (as well as f and g) and we write $l(\phi_1, \psi_1)$ when we want to emphasize this dependence. If l_1 and l_2 are two different quasiconformal homeomorphisms satisfying the above conditions, then it follows directly that l_1 and l_2 are homotopic. Thus $[X\#Y, l_1, X_1\#Y_1, \phi'] = [X\#Y, l_2, X_1\#Y_1, \phi']$.

Remark 4.6.1. Although it is probably clear that such l exist we outline a method of construction based on the ideas of Section 4.3. On X we can choose $l = f$. Away from an annular neighborhood of the boundary $\psi_1^{-1}(S^1)$ in Y we choose $l = g$. Then the problem can be reduced to constructing a quasiconformal homeomorphism from an annulus to itself such that one boundary is fixed and the other is a specified map (so that l becomes well defined on S). This quasiconformal homeomorphism is analogous to the map H constructed in Section 4.3 (see also Diagram 4.2). It is simpler in the case at hand as the shape of the boundary is not changing. For any map of the outer circle to itself, the map H will be quasiconformal.

In summary, we have constructed a sewing map

$$W : \widetilde{T}_B(X) \times \widetilde{T}_B(Y) \longrightarrow \widetilde{T}_B(X\#Y)$$

by $([X, f, X_1, \phi], [X, g, X_2, \psi]) \mapsto [X\#Y, l, X_1\#Y_1, \phi']$. This map depends on the choice of local coordinates on the reference surfaces X and Y.

Finally we can prove that *the sewing operation is holomorphic on the Teichmüller space of rigged surfaces.*

Theorem 4.6.2 (holomorphicity of sewing). *The sewing map*

$$W : \widetilde{T}_B(X) \times \widetilde{T}_B(Y) \longrightarrow \widetilde{T}_B(X\#Y)$$

is holomorphic.

Proof. It is just the matter of putting everything we already have together. The basic idea is that Schiffer variation and variation of the local coordinates together give an open neighborhood in $\widetilde{T}_B(\Sigma)$. By the infinite-dimensional version of Hartogs' theorem (Theorem 2.2.5) it is enough to prove holomorphicity in the Teichmüller space direction and the local coordinate direction separately.

We will work in a neighborhood of the points $[X, f, X^1, \phi]$ and $[Y, g, Y^1, \psi]$ where the sewing is performed using ϕ_1 and ψ_1 as above. After sewing we have an element $[X\#Y, l, X_1\#Y_1, \phi']$.

We first consider variations in the local coordinates. Let ϕ_1^s and ψ_1^t be local coordinates that depend holomorphically on s and t respectively. Let $X_s^1\#Y_t^1$ be the surface obtained from sewing using ϕ_1^s and ψ_1^t. From Theorem 4.5.2 and Proposition 4.5.7 we have a map

$$\Phi_{s,t} : X^1\#Y^1 \to X_s^1\#Y_t^1$$

that depends holomorphically on s and t. Now,

$$W([X, f, X_s^1, (\phi_1^s, \phi_2, \ldots, \phi_m)], [Y, g, Y_t^1, (\psi_1^t, \psi_2, \ldots, \psi_n)]) = [X\#Y, \Phi_{s,t} \circ l, X_s^1\#Y_t^1, \phi']$$

because $\Phi_{s,t} \circ l$ can be chosen for $l(\phi_1^s, \psi_1^t)$.

To show W is holomorphic we need to use the definition of the complex structure on $\widetilde{T}(X\#Y)$. Recall from Section 3.3.3 that charts on $\widetilde{T}(X\#Y)$ are defined using

Schiffer variation. Also recall that two such neighborhoods were compared by mapping the holomorphic families produced by Schiffer variation into the universal space $V(G)$. The same idea can be used in the present case. By Proposition 4.5.7 we know that the family of surfaces $X_s^1 \# Y_t^1$ is a marked holomorphic family with marking map $\Phi_{s,t} \circ l : X\#Y \to X_s^1\#Y_t^1$. The composition with the fixed map l does not effect this result.

By the universality of $V(G)$ there exists a morphism (that is, a holomorphic fiber space map) from the family of surfaces $X_s^1\#Y_t^1$ into $V(G)$. Because the map from the Schiffer family into $V(G)$ is also holomorphic we get a holomorphic map from the family $X_s^1\#Y_t^1$ to the Schiffer family.

To summarize we write the sequence of maps used to express W in terms of local coordinates. This is not completely precise but it illustrates the idea.

$$(s,t) \longmapsto (\phi_1^s, \psi_1^t) \longmapsto ([X, f, X_s^1, (\phi_1^s, \phi_2, \ldots, \phi_m)], [Y, g, Y_t^1, (\psi_1^t, \psi_2, \ldots, \psi_n)] \longmapsto$$
$$\overset{W}{\longmapsto} [X\#Y, \Phi_{s,t} \circ l, X_s^1\#Y_t^1, \phi'] \longrightarrow$$
$$\longrightarrow V(G) \longrightarrow \text{Schiffer family} \mapsto (\epsilon_1, \epsilon_2)$$

What we have proved is that ϵ_1 and ϵ_2 depend holomorphically on s and t.

So far we have not discussed what happens to the local coordinates ϕ'. The details are exactly as in the proof of Theorem 3.3.9, where the complex manifold structure of $\widetilde{T}(\Sigma)$ is proved. So we just outline the idea.

For fixed (s,t), let $\sigma_{s,t}$ be the biholomorphism from $X_s^1\#Y_t^1$ to the corresponding surface in the Schiffer family. (This maps is analogous to the map σ_{ϵ_1} in the proof of Theorem 3.3.9). As s and t vary, the family of maps $\sigma_{s,t}$ gives the morphism between the holomorphic family of surfaces $X_s^1\#Y_t^1$ and the Schiffer family. So $\sigma_{s,t}$ depends holomorphically on s and t.

In mapping to the Schiffer family, the local coordinates ϕ' get composed with $(\sigma_{s,t})^{-1}$. So in the Schiffer family the local coordinates are $\phi' \circ (\sigma_{s,t})^{-1}$, and certainly these depend holomorphically on s and t. Combining this with the above result we see that W depends holomorphically on s and t.

We now turn to the case where the local coordinates are fixed and the variation is in the Teichmüller space direction. It is enough to consider Schiffer variation on just one surface, say X, that produces a neighborhood with elements $[X, \nu^\epsilon \circ f, X_1^\epsilon, \phi]$. The key point is that the operations of Schiffer variation and sewing commute, as long as the Schiffer variation is performed away from the boundary to be sewn. That is, $(X_1 \# Y_1)^\epsilon = X_1^\epsilon \# Y$. From this it follows that

$$W([X, \nu^\epsilon \circ f, X_1^\epsilon, \phi], [Y, g, Y_1, \psi]) = [X\#Y, \nu^\epsilon \circ l, X_1^\epsilon \# Y, \phi'],$$

where on the right hand side ν^ϵ is extended to Y by the identity. Since the complex structure on $\widetilde{T}_B(X\#Y)$ is defined by Schiffer variation, the map

$$[X\#Y, \nu^\epsilon \circ l, X_1^\epsilon \# Y, \phi'] \mapsto \epsilon$$

is, by definition, a local coordinate. Thus in local coordinates W is

$$\epsilon \longmapsto [X, \nu^\epsilon \circ f, X_1^\epsilon, \phi] \longmapsto [X\#Y, \nu^\epsilon \circ l, X_1^\epsilon \# Y, \phi'] \longmapsto \epsilon$$

which is certainly holomorphic.

□

So far only the sewing of disjoint surfaces has been considered. In conformal field theory it is also necessary to consider the self-sewing of surfaces.

Let $\Sigma_{1,2}$ be a Riemann surface of conformal type (g, n) with punctures p_1 and p_2 oppositely-oriented. For $i = 1, 2$, let $\phi_i^{t_i}$ be holomorphic families of local coordinates at the punctures p_i. By cutting out $(\phi_i^{t_i})^{-1}(D)$ we produce a surface with parametrized boundaries, say S_1 and S_2. Let $\Sigma_{1\#2}^{t_1,t_2}$ be the surface obtained by sewing S_1 to S_2 using the parametrizations $(\phi_i^{t_i})^{-1}$.

We can now formulate a Corollary of Theorem 4.5.2.

Corollary 4.6.3. *There exists quasiconformal homeomorphisms*

$$\Phi_{t_1,t_2} : \Sigma_{1,2} \longrightarrow \Sigma_{1\#2}^{t_1,t_2}$$

and moreover $\mu(\Phi_{t_1,t_2})$ depends holomorphically on t.

Idea of proof. There are two ways to prove this. One is to repeat everything that was needed in the proof of Theorem 4.5.2 for the self-sewing case. There are no obstacles to doing this.

The second way is to consider self-sewing as a regular sewing by cutting along an extra curve (or curves) that separate the surface. On this additional curve we perform the sewing using fixed parametrizations. In this case Theorem 4.5.2 needs to be generalized to the case of performing two sewing operations simultaneously. This is also straightforward as the important constructions all take place in an annular neighborhood of the boundaries. □

An analogy to Theorem 4.6.2 can also be formulated in the self-sewing case. Let $[\Sigma_{1,2}, f, \Sigma'_{1,2}, \phi]$ be an element of $\widetilde{T}_B(\Sigma_{1,2})$.

A quasiconformal homeomorphism $l : \Sigma_{1\#2} \to \Sigma'_{1\#2}$ can be constructed as in the discussion preceding Theorem 4.6.2. Then the map

$$W : \widetilde{T}_B(\Sigma_{1,2}) \longrightarrow \widetilde{T}_B(\Sigma_{1\#2}),$$

given by

$$[\Sigma_{1,2}, f, \Sigma'_{1,2}, (\phi_1, \ldots, \phi_n)] \longmapsto [\Sigma_{1\#2}, l, \Sigma'_{1,2}, (\phi_3, \ldots, \phi_n)]$$

is well defined.

Corollary 4.6.4. *The map W is holomorphic.*

Idea of proof. The proof of Theorem 4.6.2 can be adapted without difficulty. □

Chapter 5

Determinant Line Bundle over Rigged Teichmüller Space

On a Riemann surface with analytically parametrized boundary there is a special Fredholm operator which is the direct sum of the $\bar{\partial}$ operator and a certain projection operator. The *determinant* of this operator is one-dimensional complex vector space, which we call the *determinant line* associated to the Riemann surface. The goal of this chapter is to prove that these determinant lines form a holomorphic vector bundle over the moduli space of Riemann surfaces with analytically parametrized boundaries. Furthermore we show that the canonical isomorphism of determinant lines corresponding to the sewing operation is holomorphic.

The basic idea will be to give a pants (3-holed spheres) decomposition and use the genus-zero trivialization of the determinant line on each pair of pants. Local trivialization will be obtained by using Schiffer variation to choose pants decompositions in a standard way in a neighborhood of any point in Teichmüller space. To compare two trivialization we will use the result of Hatcher and Thurston [24] that any two pants decompositions are related by a sequence of two types of basic moves called *elementary moves*. Associativity of the sewing operation will be proved and this will allow us to express the transition function in terms of these moves. Elementary moves involve only 4-holed spheres and 1-holed tori (the so-called *level one surfaces*). We already know that the sewing operation is analytic in genus-zero. Once we establish this for genus-one then the holomorphicity of the general transition functions will follow.

Remark 5.0.5. The ideas just outlined were partially inspired by *Grothendieck's reconstruction principle* as described in Luo [40]. Although the analogy is not completely precise, the idea that the *higher-level* structure is determined by the *level 0* and *level 1* surfaces is compelling. (Level zero surfaces are 3-holed spheres.)

The structure of this chapter is as follows. An overview of determinant lines of Riemann surfaces with analytically parametrized boundaries and the known holomorphicity results in genus-zero will be given in Section 5.1. Construction of the sewing isomorphism and the proof of its crucial associativity properties for surfaces of arbitrary genus appear in Sections 5.2 and 5.2.2. The only significant work needed in these generalizations to higher-genus is the proof that the map $\Delta : \mathrm{Hol}(\Sigma_1 \sqcup \Sigma_2) \to \Omega^0(S)$ in surjective. Section 5.2.3 is devoted to this proof, while Appendix B contains the prerequisite material on Cauchy-type kernels and the Plemelj-Sokhotski formula.

A holomorphic bundle structure for determinant lines of genus-one surfaces is given in section 5.3. One of the elementary moves involves a torus with boundary obtained from the self-sewing of a sphere. In Section 5.4 we prove that this self-sewing operation in holomorphic.

Section 5.5.1 describes the result of Hatcher and Thurston that any two isotopy classes of pants decompositions can be joined by a sequence of elementary moves. By a simple argument using some classical results in topology we show in Section 5.5.2 that the elementary moves can be used to join any two fixed decompositons, not just their isotopy classes.

Finally, in Section 5.6 a trivialization for the determinant lines is given and the holomorphicity of resulting transition functions is proved. Thus we conclude that the determinant lines form a holomorphic line bundle over the rigged Teichmüller space.

Section 5.7 achieves one of our fundamental goals. Here we prove that the sewing isomorphism of determinant lines bundles is holomorphic.

To pass from the Teichmüller space to the moduli space, an action of the mapping class group on the determinant line bundle must be defined. This is done in Section 5.8 where it is then shown that the determinant lines form a holomorphic line bundle over the moduli space.

5.1 Introduction to determinant lines

This section contains a brief account of the determinant lines associated to a Riemann surfaces with analytically parametrized boundaries. Important definitions and properties as well as the genus-zero results are given. The material has been taken directly from Huang [30, Appendix D], where a detailed exposition can be found.

A $\mathbb{Z}_2$*-graded line* is a one-dimensional vector space together with a an element of $\mathbb{Z}_2$ called the *degree*. Let V be an n-dimensional vector space. The *determinant line* $\mathrm{Det}(V)$ of V is defined to be $\bigwedge^n(V)$, the top exterior power of V, with degree n modulo $2\mathbb{Z}$. Given two graded lines L_1 and L_2, $L_1 \otimes L_2$ with degree $\deg(L_1 \otimes L_2) = \deg(L_1) + deg(L_2)$ is a graded line. This defines a tensor product operation. The isomorphism $L_1 \otimes L_2 \to L_2 \otimes L_1$ given by

$$v_1 \otimes v_2 \mapsto (-1)^{\deg(v_1)\deg(v_2)} v_2 \otimes v_1$$

defines a natural isomorphism from the tensor product functor to itself. Note that this natural isomorphism is the one we use to swap the order of spaces in a tensor product.

Let $F : H_1 \to H_2$ be a Fredholm operator between two Hilbert spaces. By definition $\mathrm{Ker}(F)$ and $\mathrm{Coker}(F)$ are finite dimensional.

Definition 5.1.1. The *determinant line of the Fredholm operator F* is the graded line

$$\mathrm{Det}(F) = \mathrm{Det}(\mathrm{Ker}\, F)^* \otimes \mathrm{Det}(\mathrm{Coker}\, F)$$

In Section 4.1, Riemann surfaces with analytically parametrized boundaries and the sewing operation were defined. Let Σ be a Riemann surface with analytically parametrized boundary and recall that the orientation of the boundary depends on whether the parametrization extends inside or outside the unit circle.

Smooth functions on the circle can be split into their positive and negative Fourier components. Let $\Omega^0(S^1)$ be the space of smooth functions on S^1. We define the spaces

$$\Omega^0_{\geq 0}(S^1) = \left\{ f \in \Omega^0(S^1) \,|\, f = \sum_{n=0}^{\infty} a_n \mathrm{e}^{in\theta} \right\},$$

and

$$\Omega^0_{<0}(S^1) = \left\{ f \in \Omega^0(S^1) \,|\, f = \sum_{n=-1}^{-\infty} a_n \mathrm{e}^{in\theta} \right\}.$$

In Huang [30] these spaces are called $\Omega^0_\pm(S^1)$. Our change of notation is to facilitate the introduction of several related spaces is Section 5.3. Note that there is a direct sum decomposition

$$\Omega^0(S^1) = \Omega^0_{<0}(S^1) \oplus \Omega^0_{\geq 0}(S^1).$$

Let C be a boundary component of Σ and let $\phi : S^1 \to C$ be an analytic parametrization of C. Denote by $\Omega^0(C)$ the space of smooth functions on C. The map $\Omega^0(C) \to \Omega^0(S^1)$ given by $g \mapsto g \circ \phi$ is an isomorphism.

Let $\partial\Sigma = \cup_{i=1}^n C_i^{\epsilon_i}$, where $\epsilon_i = \pm$ depending on whether the boundary component is positively or negatively oriented. For $i = 1, \ldots, n$, the parametrizations of C_i will be written ϕ_i. If $\epsilon_i = +$ ($\epsilon_i = -$) we define

$$\Omega^0_+(C_i^{\epsilon_i}) = \{g \in \Omega^0(C_i^{\epsilon_i}) \,|\, g \circ \phi_i \in \Omega^0_{\geq 0}(S^1),\ (\Omega^0_{<0}(S^1))\}$$

and

$$\Omega^0_-(C_i^{\epsilon_i}) = \{g \in \Omega^0(C_i^{\epsilon_i}) \,|\, g \circ \phi_i \in \Omega^0_{<0}(S^1),\ (\Omega^0_{\geq 0}(S^1))\}.$$

Let

$$\Omega^0_\pm(\partial\Sigma) = \bigoplus_{i=1}^n \Omega^0_{\pm\epsilon_i}(C_i^{\epsilon_i})$$

and let $\Omega^{0,1}(\Sigma)$ be the space of $(0,1)$ forms on Σ. The projection operator from $\Omega^0(\partial\Sigma)$ to $\Omega^0_+(\partial\Sigma)$ will be denoted by pr. It can be shown that the operator

$$\bar\partial \oplus \mathrm{pr} : \Omega^0(\Sigma) \longrightarrow \Omega^{0,1}(\Sigma) \oplus \Omega^0_+(\partial\Sigma)$$

is a Fredholm operator after the spaces are suitably completed to Hilbert spaces. Instead of working directly with $\bar\partial \oplus \mathrm{pr}$, we introduce the Fredholm operator

$$\pi_\Sigma : \mathrm{Hol}(\Sigma) \longrightarrow \Omega^0_+(\partial\Sigma)$$

defined by restricting pr to $\mathrm{Hol}(\Sigma)$. There is a canonical isomorphism from $\mathrm{Det}(\partial \oplus \mathrm{pr})$ to $\mathrm{Det}(\pi_\Sigma)$ (see [30, Proposition D.3.3]).

Definition 5.1.2. The *determinant line,* Det_Σ*, over* Σ is defined by

$$\mathrm{Det}_\Sigma = \mathrm{Det}(\pi_\Sigma).$$

The map $\Sigma \mapsto \mathrm{Det}_\Sigma$ defines a functor from the category of Riemann surfaces with analytically parametrized boundaries to the category of one-dimensional complex vector spaces.

We now define the canonical isomorphism associated to sewing and discuss its associativity property. Let Σ_1 and Σ_2 be a Riemann surfaces with oppositely-oriented boundary components S_1 and S_2. Let $\Sigma_1 \# \Sigma_2$ denote the surface obtained by sewing S_1 to S_2 and let S be the curve on $\Sigma_1 \# \Sigma_2$ corresponding to S_1 or S_2. We want to construct a canonical isomorphism

$$\ell_{\Sigma_1,\Sigma_2} : \mathrm{Det}_{\Sigma_1 \otimes \Sigma_2} \longrightarrow \mathrm{Det}_{\Sigma_1 \# \Sigma_2}\,. \tag{5.1}$$

We define the maps

$$\Delta_{\Sigma_1,\Sigma_2} : \mathrm{Hol}(\Sigma_1 \sqcup \Sigma_2) \to \Omega^0(S)$$

by

$$f \longmapsto f|_{S_1} - f|_{S_2}.$$

and

$$\bar{\pi}_{\Sigma_1,\Sigma_2} : \mathrm{Hol}(\Sigma_1 \sqcup \Sigma_2) \to \Omega^0_+(\partial(\Sigma_1 \# \Sigma_2)) \oplus \Omega^0(S) \tag{5.2}$$

by

$$f \longmapsto \pi_{\Sigma_{1\#2}}(f), \Delta_{\Sigma_1,\Sigma_2}(f).$$

In the case of genus-zero case, there is a commutative diagram (see Huang [30, Diagram (D.4.1)])

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hol}(\Sigma_1 \# \Sigma_2) & \longrightarrow & \mathrm{Hol}(\Sigma_1 \sqcup \Sigma_2) & \xrightarrow{\Delta_{\Sigma_1,\Sigma_2}} & \Omega^0(S) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi_{\Sigma_1\#\Sigma_2}} & & \downarrow{\scriptstyle \bar{\pi}_{\Sigma_1,\Sigma_2}} & & \downarrow{\scriptstyle I} & & \\
0 & \longrightarrow & \Omega^0_+(\partial(\Sigma_1 \# \Sigma_2)) & \longrightarrow & \Omega^0_+(\partial(\Sigma_1 \# \Sigma_2)) \oplus \Omega^0(S) & \longrightarrow & \Omega^0(S) & \longrightarrow & 0.
\end{array} \tag{5.3}$$

Showing that Δ_{S_1,S_2} is surjective is the non-trivial part. In higher-genus the surjectivity is proved in Section 5.2.3.

We now want to show that $\mathrm{Det}\,\pi_{\Sigma_1 \sqcup \Sigma_2}$ and $\mathrm{Det}\,\bar{\pi}_{\Sigma_1,\Sigma_2}$ are canonically isomorphic. By identifying $\Omega^0_-(S_1) \oplus \Omega^0_+(S_2)$ with $\Omega^0(S)$ the maps $\pi_{\Sigma_1 \sqcup \Sigma_2}$ and $\bar{\pi}_{\Sigma_1,\Sigma_2}$ can be thought of as having the same codomain. The difference of these operators is

$$(\pi_{\Sigma_1 \sqcup \Sigma_2} - \bar{\pi}_{\Sigma_1,\Sigma_2})(f) = (f|_{S_1})_- - (f|_{S_2})_+.$$

It is proved in [30, Lemma D.4.3] that this difference is a trace-class operator and thus there is a canonical isomorphism between $\mathrm{Det}(\pi_{\Sigma_1 \sqcup \Sigma_2})$ and $\mathrm{Det}(\bar{\pi}_{\Sigma_1,\Sigma_2})$. Composing this isomorphism with the isomorphism from $\mathrm{Det}\,\bar{\pi}_{\Sigma_1,\Sigma_2}$ to $\mathrm{Det}\,\pi_{\Sigma_1 \# \Sigma_2}$, we obtain a canonical isomorphism

$$\ell_{\Sigma_1,\Sigma_2} : \mathrm{Det}_{\Sigma_1 \otimes \Sigma_2} \longrightarrow \mathrm{Det}_{\Sigma_1 \# \Sigma_2}$$

as desired. We call this the *canonical isomorphism associated to sewing.*

The sewing isomorphism satisfies a crucial associativity property summarized by the following commutative diagram (see [30, Diagram D.4.4]).

$$\begin{array}{ccc} \mathrm{Det}_{\Sigma_1} \otimes \mathrm{Det}_{\Sigma_2} \otimes \mathrm{Det}_{\Sigma_3} & \xrightarrow{\ell_{\Sigma_1,\Sigma_2} \otimes I} & \mathrm{Det}_{\Sigma_1 \# \Sigma_2} \otimes \mathrm{Det}_{\Sigma_3} \\ \Big\downarrow {\scriptstyle I \otimes \ell_{\Sigma_2,\Sigma_3}} & & \Big\downarrow {\scriptstyle \ell_{\Sigma_1 \# \Sigma_2, \Sigma_3}} \\ \mathrm{Det}_{\Sigma_1} \otimes \mathrm{Det}_{\Sigma_2 \# \Sigma_3} & \xrightarrow{\ell_{\Sigma_1, \Sigma_2 \# \Sigma_3}} & \mathrm{Det}_{\Sigma_1 \# \Sigma_2 \# \Sigma_3} \end{array} \tag{5.4}$$

We now informally summarize the genus-zero holomorphicity results stated in [30, Propositions D.4.7 and D.4.8].

Theorem 5.1.1. *The determinant lines form a trivial holomorphic line bundle over the moduli space of genus-zero Riemann surfaces with analytically parametrized boundaries. There is a canonical holomorphic flat connection on the determinant line bundle. The sewing isomorphism is holomorphic.*

5.2 Sewing Isomorphism

A crucial property of the determinant line bundle is the sewing isomorphism introduced in Section 5.1. Our long-term goal is to prove that this isomorphism is holomorphic. The aim of this is section is to construct this isomorphism and prove its assoicativity property.

In Section 5.1 we outlined the construction in Huang [30], of the sewing isomorphism and its associativity property for genus-zero surfaces. In this case the sewing is always between different surfaces, so self-sewing is not considered. Fortunately, the methods used in [30] can be applied to higher-genus surfaces and self-sewing. The only substantial work to be done is proving that the map $\Delta : \mathrm{Hol}(\Sigma_1 \sqcup \Sigma_2) \to \Omega^0(S)$ is onto. This problem is addressed separately in Section 5.2.3.

Remark 5.2.1. Actually the higher-genus results for regular sewing follow immediately once Δ is proved to be surjective. The construction of the sewing isomorphism and the proof of its associativity given in [30] do not need to be changed at all.

5.2.1 Construction of the self-sewing isomorphism

The notation and structure of this section will closely follow Section 5.1. See also Huang, [30, Appendix D.4].

Recall that the determinant line of a Riemann surface can be defined by the determinant of the Fredholm operator

$$\pi_\Sigma : \mathrm{Hol}(\Sigma) \longrightarrow \Omega^0_+(\partial\Sigma)$$

given by the composition of restriction to the boundary and projection to the positive part. For $f \in \Omega^0(\Sigma)$, let f_+ be the component of f in $\Omega^0_+(\Sigma)$.

Let $\Sigma_{1,2}$ be a Riemann surface with oppositely-oriented boundary components S_1 and S_2. Let $\Sigma_{1\#2}$ denote the surface obtained by sewing S_1 to S_2 and let S be the curve on $\Sigma_{1\#2}$ corresponding to S_1 or S_2.

We want to construct a canonical isomorphism

$$\ell_{\Sigma_{1,2};S_1,S_2} : \mathrm{Det}_{\Sigma_{1,2}} \longrightarrow \mathrm{Det}_{\Sigma_{1\#2}} \tag{5.5}$$

Remark 5.2.2. Sewing of distinct surfaces involves considering the disjoint union $\Sigma_1 \sqcup \Sigma_2$. In our case this corresponds to the surface $\Sigma_{1,2}$.

Remark 5.2.3. The subscript $\Sigma_{1,2}$ of Δ and ℓ specifies the surface and the subscript S_1, S_2 specifies the curves to be sewn.

In this self-sewing case, the Δ map

$$\Delta_{\Sigma_{1,2};S_1,S_2} : \mathrm{Hol}(\Sigma_{1,2}) \to \Omega^0(S)$$

is defined by

$$f \longmapsto f|_{S_1} - f|_{S_2}.$$

and we define

$$\bar{\pi}_{\Sigma_{1,2};S_1,S_2} : \mathrm{Hol}(\Sigma_{1,2}) \to \Omega^0_+(\partial(\Sigma_{1\#2})) \oplus \Omega^0(S) \tag{5.6}$$

by

$$f \longmapsto \pi_{\Sigma_{1\#2}}(f), \Delta_{\Sigma_{1,2};S_1,S_2}(f).$$

When the underlying surface is clear we simplify notation and write Δ_{S_1,S_2} and $\bar{\pi}_{S_1,S_2}$ instead of $\Delta_{\Sigma_{1,2};S_1,S_2}$ and $\bar{\pi}_{\Sigma_{1,2};S_1,S_2}$.

The commutative diagram corresponding to Diagram 5.3 is

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hol}(\Sigma_{1\#2}) & \longrightarrow & \mathrm{Hol}(\Sigma_{1,2}) & \xrightarrow{\Delta_{S_1,S_2}} & \Omega^0(S) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi_{\Sigma_{1\#2}}} & & \downarrow{\scriptstyle \bar{\pi}_{S_1,S_2}} & & \downarrow{\scriptstyle I} & & \\ 0 & \longrightarrow & \Omega^0_+(\partial(\Sigma_{1\#2})) & \longrightarrow & \Omega^0_+(\partial(\Sigma_{1\#2})) \oplus \Omega^0(S) & \longrightarrow & \Omega^0(S) & \longrightarrow & 0. \end{array} \tag{5.7}$$

In Section 5.2.3 it will be shown that Δ_{S_1,S_2} is surjective. Accepting this for now, the exactness of the rows follows easily as in [30]. So by the Snake lemma ([30], D.2.3), $\mathrm{Det}\,\bar{\pi}_{S_1,S_2}$ and $\mathrm{Det}\,\pi_{\Sigma_{1\#2}}$ are canonically isomorphic

We now want to produce a canonical isomorphism bewteen $\mathrm{Det}\,\pi_{\Sigma_{1,2}}$ and $\mathrm{Det}\,\bar{\pi}_{S_1,S_2}$. By identifying $\Omega^0_-(S_1) \oplus \Omega^0_+(S_2)$ with $\Omega^0(S)$ the maps $\pi_{\Sigma_{1,2}}$ and $\bar{\pi}_{S_1,S_2}$ can be thought of as having the same codomain. The difference of these operators is

$$(\pi_{\Sigma_{1,2}} - \bar{\pi}_{S_1,S_2})(f) = (f|_{S_1})_- - (f|_{S_2})_+.$$

The proof of Lemma D.4.3 in [30], that such a difference is a trace-class operator, involves only an annular neighborhood of the curve S. The fact that in our case the two curves S_1 and S_2 are on the same surface makes no difference and the proof can be applied directly. Thus $\mathrm{Det}\,\pi_{\Sigma_{1,2}} \simeq \mathrm{Det}\,\bar{\pi}_{S_1,S_2}$.

Composing this isomorphism with the isomorphism from $\mathrm{Det}\,\bar{\pi}_{S_1,S_2}$ to $\mathrm{Det}\,\pi_{\Sigma_{1\#2}}$, we obtain a canonical isomorphism

$$\ell_{\Sigma_{1,2};S_1,S_2} : \mathrm{Det}_{\Sigma_{1,2}} \longrightarrow \mathrm{Det}_{\Sigma_{1\#2}}$$

as desired. We call this the *canonical isomorphism associated to self-sewing.*

5.2.2 Self-sewing associativity

There are two types of associativity that must be considered. The first is where one of the sewing operations is self-sewing. The second is where both of the sewing operations are self-sewing. These cases are covered by the diagrams considered in Kriz [37], where the categorical structure (coherence) of sewing and disjoint union are discussed.

The associativity for sewing genus-zero surfaces is proved in Huang [30, Theorem D.4.4]. With only some notational changes this proof can be used almost verbatim in our cases. The only significantly new work to be done is in the proof of the surjectivity of the Δ maps which is addressed in Section 5.2.3. Instead of repeating the several page argument in [30] we outline the changes that must be made.

Theorem 5.2.4. *The sewing isomorphism is associative in all cases where the resultant sewn surface has at least one boundary component.*

The rest of this section is dedicated to the outline of the proof of this theorem and the precise statements of the associativity.

Remark 5.2.5. The restriction to the case of at least one boundary component comes from the fact that Δ is only surjective in these cases. Moreover, for surface with no boundary components the definition of the determinant line must be different and we do not discuss this issue.

Remark 5.2.6. There are a couple of misprints in the proof of Theorem D.4.4 in [30]. On page 248, the second equation in the second paragraph should be

$$\bar{\pi}_{\Sigma_1\#\Sigma_2,\Sigma_3} + (\pi_{\Sigma_1\#\Sigma_2} \oplus \pi_{\Sigma_3} - \bar{\pi}_{\Sigma_1\#\Sigma_2,\Sigma_3}) = \pi_{\Sigma_1\#\Sigma_2} \oplus \pi_{\Sigma_3}$$

and the second expression below this should change correspondingly to

$$\pi_{\Sigma_1\#\Sigma_2} \oplus \pi_{\Sigma_3} - \bar{\pi}_{\Sigma_1\#\Sigma_2,\Sigma_3}.$$

In the middle of page 249 the expression should change to

$$\Omega^0_+(\partial(\Sigma_1\#\Sigma_2))\#\Sigma_3 \oplus \Omega^0(S_{23})$$

and the words after Equation (D.4.13) should read "From (D.4.12) and (D.4.13)".

Let Σ_1 be Riemann surfaces whose boundary contains a component S_1. Let $\Sigma_2^{3,4}$ be a Riemann surface whose boundary contains components S_2, S_3 and S_4. Assume that S_1 and S_2 can be sewn, and we denote the sewn surface by $\Sigma_1 \# \Sigma_2^{3,4}$. Further assume that S_3 and S_4 can be sewn, and we denote the sewn surface by $\Sigma_2^{3\#4}$.

To make the comparison between this case and the case for regular sewing (see Section 5.1) easier, we list the relationships between the surfaces.

$$\Sigma_1 \longleftrightarrow \Sigma_1 \quad , \quad \Sigma_2 \sqcup \Sigma_3 \longleftrightarrow \Sigma_2^{3,4} \quad , \quad \Sigma_1 \# \Sigma_2 \# \Sigma_3 \longleftrightarrow \Sigma_1 \# \Sigma_2^{3\#4}$$

Associativity of the sewing operation corresponds to the commutativity of the following diagram.

$$\begin{array}{ccc} \mathrm{Det}_{\Sigma_1} \otimes \mathrm{Det}_{\Sigma_2^{3,4}} & \xrightarrow{\ell_{\Sigma_1, \Sigma_2^{3,4}}} & \mathrm{Det}_{\Sigma_1 \# \Sigma_2^{3,4}} \\ {\scriptstyle I \otimes \ell_{\Sigma_2^{3,4}; S_3, S_4}} \downarrow & & \downarrow {\scriptstyle \ell_{\Sigma_1 \# \Sigma_2^{3,4}; S_3, S_4}} \\ \mathrm{Det}_{\Sigma_1} \otimes \mathrm{Det}_{\Sigma_2^{3\#4}} & \xrightarrow{\ell_{\Sigma_1, \Sigma_2^{3\#4}}} & \mathrm{Det}_{\Sigma_1 \# \Sigma_2^{3\#4}} \end{array} \tag{5.8}$$

This diagram is equivalent to Kriz [37, Diagram 2.6]. For regular sewing the analogous associativity is expressed in Diagram 5.4.

We now explain how the proof of the commutativity of Diagram 5.8 can be obtained by emulating the proof of Theorem D.4.4 in Huang [30]. Since the sewing isomorphism is constructed from a composition of two maps, the commutativity becomes equivalent to the commutativity of four smaller diagrams.

Let S_{12} be the curve on $\Sigma_1 \# \Sigma_2^{3,4}$ corresponding to S_1 or S_2, and let S_{34} be the curve on $\Sigma_2^{3\#4}$ corresponding to S_3 or S_4. The map corresponding to $\bar{\pi}_{\Sigma_1, \Sigma_2, \Sigma_3}$ is

$$\bar{\pi}_{\Sigma_1, \Sigma_2^{3,4}; S_3, S_4} : \mathrm{Hol}(\Sigma_1 \sqcup \Sigma_2^{3,4}) \to \Omega_+^0(\partial(\Sigma_1 \# \Sigma_2^{3\#4})) \oplus \Omega^0(S_{12}) \oplus \Omega^0(S_{34})$$

which is defined by

$$f \longmapsto \pi_{\Sigma_1 \# \Sigma_2^{3\#4}}(f), \Delta_{\Sigma_1, \Sigma_2^{3,4}}(f), \Delta_{\Sigma_2^{3,4}; S_3, S_4}(f)$$

Some of the correspondences between the spaces and maps in [30, Diagram D.4.7], and the ones in our self-sewing case, are listed below.

$$\mathrm{Hol}(\Sigma_1\#\Sigma_2)\oplus\mathrm{Hol}(\Sigma_3)\longleftrightarrow\mathrm{Hol}(\Sigma_1\#\Sigma_2^{3,4})$$

$$\pi_{\Sigma_1\#\Sigma_2}\oplus\pi_{\Sigma_3}\longleftrightarrow\pi_{\Sigma_1\#\Sigma_2^{3,4}}$$

$$\bar{\pi}_{\Sigma_1,\Sigma_2}\oplus\pi_{\Sigma_3}\longleftrightarrow\bar{\pi}_{\Sigma_1,\Sigma_2^{3,4}}$$

Using these, and corresponding changes for the other diagrams, it is just a long but straightforward exercise to convert the proof in [30] to our case. This concludes our discussion of associativity for the case of one self-sewing and one regular sewing.

For the case of when both the sewing operations are self-sewing, similar changes need to be made. Let $\Sigma_{1,2}^{3,4}$ be a Riemann surface whose boundary contains components S_1, S_2, S_3 and S_4. Assume that S_1 and S_2 can be sewn and denote the sewn surface by $\Sigma_{1\#2}^{3,4}$. Similarly we assume that S_3 and S_4 can be sewn and denote the sewn surface by $\Sigma_{1,2}^{3\#4}$.

The diagram corresponding to Diagram 5.4 (also see Kriz [37, Diagram 2.7]) is

$$\begin{array}{ccc} \mathrm{Det}_{\Sigma_{1,2}^{3,4}} & \xrightarrow{\ell_{\Sigma_{1,2}^{3,4};S_1,S_2}} & \mathrm{Det}_{\Sigma_{1\#2}^{3,4}} \\ \downarrow{\scriptstyle \ell_{\Sigma_{1,2}^{3,4};S_3,S_4}} & & \downarrow{\scriptstyle \ell_{\Sigma_{1\#2}^{3,4};S_3,S_4}} \\ \mathrm{Det}_{\Sigma_{1,2}^{3\#4}} & \xrightarrow{\ell_{\Sigma_{1,2}^{3\#4};S_1,S_2}} & \mathrm{Det}_{\Sigma_{1\#2}^{3\#4}} \end{array} \quad . \tag{5.9}$$

Associativity of the sewing operation corresponds to the commutativity of the above diagram. The proof of commutativity can be adapted in a way similar to above. We omit the details which are again a long, but straightforward, exercise.

We can now conclude that in any genus the sewing isomorphism is associative for any combination of regular sewing and self-sewing. This completes our outline of the proof of Theorem 5.2.4.

5.2.3 The Δ-map for higher-genus

An important step in the construction of the sewing isomorphism and proving its associativity properties uses the surjectivity of the maps $\Delta_{\Sigma_1,\Sigma_2}:\mathrm{Hol}(\Sigma_1\sqcup\Sigma_2)\to\Omega^0(S)$

and $\Delta_{\Sigma_{1,2}} : \mathrm{Hol}(\Sigma_{1,2}) \to \Omega^0(S)$. See Section 5.2, Diagram 5.7 and Section 5.2.2, as well the details provided in the proof of Theorem D.4.4 in Huang [30]. We will prove that the Δ map is onto provided that the the sewn surface has at least one boundary component. See Appendix B for background on the Plemelj-Sokhotski formula for Riemann surfaces.

Let Σ be a Riemann surface, $\gamma \subset \Sigma$ be a simple closed curve, and $g : \gamma \to \mathbb{C}$ be a smooth function. The *jump problem* is to find a holomorphic function $f \in \mathrm{Hol}(\Sigma \setminus \gamma)$ such the limiting values of $f(p)$, as p approaches the curve γ from the two different sides, satisfy $f^+ - f^- = g$. (See Appendix B and in particular Remark B.2.2 for details).

Suppose $\Sigma = \Sigma_1 \# \Sigma_2$ and let S be the curve in Σ corresponding to the sewn boundaries S_1 and S_2 on Σ_1 and Σ_2 respectively.

Remark 5.2.7. Self sewing will be discussed later. It follows easily from the case of regular sewing

Theorem 5.2.8. *If $\Sigma = \Sigma_1 \# \Sigma_2$ has at least one boundary component then the map $\Delta_{\Sigma_1,\Sigma_2} : \mathrm{Hol}(\Sigma_1 \sqcup \Sigma_2) \to \Omega^0(S)$ given by $f \mapsto f|_{S_1} - f|_{S_2}$ is surjective.*

First proof. Showing that Δ is onto is equivalent to solving the jump problem on the curve S for arbitrary $g \in \Omega^0(S)$. We will show that this is always possible when Σ has at least one boundary component.

We sew disks to the boundary of Σ using the analytic parametrizations and call the resulting compact surface Σ_c. Let $D = D_1 \cup \cdots \cup D_N$ correspond to these disk on Σ_c and note that $\Sigma_c \setminus D = \Sigma$.

For an (essentially) arbitrary choice of points $q_1, \ldots q_g$ in Σ_c, we know from Appendix B that a kernel $M(p, q)$ can be constructed such that for any $g \in \Omega^0(S)$,

$$F(q) = \int_S g(p) M(p, q) dz(p)$$

defines a single-valued meromorphic function on $\Sigma_c \backslash S$ whose only poles are at the points q_k. Actually F really defines two separate function, F_1 and F_2, on the surfaces separated by S. By Theorem B.2.1 the boundary values of F satisfy $F_1|_S - F_2|_S = F^+ - F^- = g$.

Since the q_k were arbitrary, we can choose them to be in D. Thus F is holomorphic on $\Sigma \setminus D$ and hence, we have a holomorphic solution to the jump problem. □

Remark 5.2.9. There is less direct second method of proof which we also give. This proof was conceived to try and use the idea sketched in Segal that the surjectivity of Δ follows directly from Stein manifold theory. However, I see no way of doing this without using the result of Theorem B.2.1.

Second proof. Instead of considering Σ_c we extend Σ to an open Riemann surface, Σ_o, by sewing in annuli using the boundary parametrization. We could have obtained an open surface by deleting the boundary, but the extension is used so that the solution we obtain is smooth on the boundary components of Σ.

For any choice of points q_k, the function

$$F(q) = \int_S g(p)M(p,q)dz(p)$$

gives a meromorphic solution to the jump problem. More specifically we have meromorphic functions $F_1 = F|_{\Sigma_1}$ and $F_2 = F|_{\Sigma_2}$ defined on the surfaces separated by S such that $F_1|_S - F_2|_S = g$.

The first Cousin problem can always be solved for open Riemann surfaces and we will use this fact to produce a holomorphic solution to the jump problem.

Remark 5.2.10. Open Riemann surfaces are a special case of Stein manifolds. On these manifolds the first cohomology is zero and this is what allows the first Cousin problem to be solved. See for example Fritzsche and Grauert [17, Chapter V].

For $i = 1, 2$, let $U_i = \Sigma_i \subset \Sigma_o$. Let U_0 be an annular neighborhood of S chosen to be sufficiently small so that no poles of F lie in U_0. On U_0 define the analytic function $F_0 = 0$. Now $\{U_0, U_1, U_2\}$ is an open cover of Σ and $\{F_0, F^+, F^-\}$ is a Mittag-Leffler distribution (see Forster [13, page 202]). Hence there exists a meromorphic function G on Σ such that $G - F^+$ is holomorphic on Σ_1 and $G - F^-$ is holomorphic on Σ_2. Also $G - 0$ is holomorphic on A but we do not use this fact. Let $G_1 = G - F_1$ and $G_2 = G - F_2$. These functions are holomorphic, and so on S we have $G_1 - G_2 = g$. Hence we have a holomorphic solution to the jump problem. □

For self-sewing we consider a surface $\Sigma_{1,2}$ with boundary components S_1 and S_2 that can be sewn to produce a surface $\Sigma_{1\#2}$. Let S be the curve on $\Sigma_{1\#2}$ corresponding to S_1 or S_2 after sewing.

Theorem 5.2.11. *If $\Sigma_{1\#2}$ has at least one boundary component then the map $\Delta_{\Sigma_{1,2}}$: $\mathrm{Hol}(\Sigma_{1,2}) \to \Omega^0(S)$ defined by $f \mapsto f|_{S_1} - f|_{S_2}$ is surjective.*

Proof. The idea is to reduce this problem to the case of regular sewing. We add extra curves so that together with S, the surface is separated into two pieces. Along these extra curves the jump is required to be zero. The resulting holomorphic solution to the jump problem is continuous across the additional curves and so must be holomorphic there. ☐

5.3 Determinant line bundle in genus-one

The ideas follow that of the genus-zero case given in Huang [30]. In particular we prove that the space of holomorphic functions on a genus-one surface with boundary is isomorphic to a certain space of functions on the circle.

In the higher-genus case, proving the holomorphicity of the determinant line bundle will be reduced to surfaces of genus-zero and genus-one with one boundary component. So in this section we restrict to the case of genus-one surfaces with one boundary component although the method can be used in the general case.

5.3.1 Jump problem on the torus

The jump problem on the torus with one boundary component is a special case of the jump problem considered in Section 5.2.3. Working on the torus simplifies certain things as we can work on the fundamental parallelogram in the plane and thus there is a global coordinate.

Consider a genus-one surface Σ with one puncture. Let $[\Sigma, g, \Sigma_1, \psi] \in \widetilde{T}_B(\Sigma)$ and recall that by the definition of $\widetilde{T}_B(\Sigma)$ (in Section 4.2), that the image of ψ contains the unit disk. Cutting out $\psi^{-1}(\Delta)$ gives surface with boundary parametrized by ψ^{-1} : $S^1 \to \partial\Sigma$.

For τ in the upper half-plane, let $\mathbb{T}_\tau$ be the torus represented by the fundamental parallelogram $0, \tau, 1, \tau+1$ with opposite sides identified. In the equivalence class $[\Sigma, g, \Sigma_1, \psi] \in \widetilde{T}_B(\Sigma)$ there is a representative of the form $(\Sigma, f, \mathbb{T}_\tau, \phi^{-1})$. Let $\gamma = \phi(S^1)$, and note that it is an analytic curve on $\mathbb{T}_\tau$. Take $g : \gamma \to \mathbb{C}$ to be a smooth function. Let D^- be the domain bounded by γ, and D^+ be the region outside γ.

Let z be the coordinate on $\mathbb{T}_\tau$. The holomorphic differential on $\mathbb{T}_\tau$ is simply dz, and so by Theorem B.2.1,

$$F(z) = \frac{1}{2\pi i} \int_\gamma g(p) M(p, z) dp$$

defines holomorphic functions on $D^\pm$ if and only if

$$\int_\gamma g(p) dp = 0. \tag{5.10}$$

The Plemelj-Sokhotski formula gives $F^+(z) - F^-(z) = g(z)$.

5.3.2 Holomorphic functions on the torus with boundary

The aim is to prove that the spaces of holomorphic functions on genus-one surfaces with one analytically parametrized boundary component form a holomorphic vector bundle over Teichmüller space. We will follow the genus-zero method used in Huang [30]. The basic idea is to produce an isomorphism between the space of holomorphic functions on the surface and essentially the space of smooth functions on the unit circle with either purely negative or positive Fourier coefficients, depending on the orientation of the boundary component. The only significant difference in the genus-one case is that there is a "one-dimension obstruction" to solving the jump problem on the torus as observed in Section 5.3.1. This means that the space of smooth functions on the circle must be suitably restricted. As noted in Section 5.3.1 we only need to investigate the case of a torus with one boundary component.

An important step in both the genus-zero and genus-one arguments uses the following basic lemma about vector spaces.

Lemma 5.3.1. *Let A and B be (possibly infinite-dimensional) complex vector spaces with direct sum decompositions $A = A_+ \oplus A_-$ and $B = B_+ \oplus B_-$. If there exists an* *isomorphism* $T : A \to B$ *such that* $T|_{A_-} : A_- \to B_-$ *is also an isomorphism then* A_+ *is isomorphic to* B_+ *via the map* $g \mapsto T(g)_+$.

Proof. First note that in general $T(A_+)$ is not contained in B_+. (For example if $A = B = \mathbb{R}^2$ and T is a lower-triangular matrix.)

For $g \in A$ (respectively B), let $g_\pm$ denote the projections onto $A_\pm$ (respectively $B_\pm$). The maps

$$\begin{aligned} A_+ &\longrightarrow B_+ \\ g &\longmapsto T(g)_+ \end{aligned}$$

and

$$\begin{aligned} B_+ &\longrightarrow A_+ \\ f &\longmapsto T^{-1}(f)_+ \end{aligned}$$

will be demonstrated to be the desired isomorphism and its inverse. For $g \in A_+$ we need to show that $g = T^{-1}(T(g)_+)_+$. For $f \in B_+$ we need to show that $f = T(T^{-1}(f)_+)_+$. Since the argument is identical in both cases we just show the first.

The assumption $T|_{A_-} : A_- \to B_-$ is an isomorphism implies that $T^{-1}|_{B_-} : B_- \to A_-$ is an isomorphism and so if $f \in B_-$, then $T^{-1}(f)_- = T^{-1}(f)$ and $T^{-1}(f)_+ = 0$. For arbitrary $f \in A$, we can expand the identity $g = T^{-1}(T(g))$ as

$$\begin{aligned} g &= T^{-1}(T(g)_+) + T^{-1}(T(g)_-) \\ &= \big(T^{-1}(T(g)_+)_+ + T^{-1}(T(g)_+)_-\big) + \big(T^{-1}(T(g)_-)_+ + T^{-1}(T(g)_-)_-\big) \qquad (5.11) \end{aligned}$$

Now, if $g \in A_+$, then $g = g_+$ and so $T^{-1}(T(g)_+)_- + T^{-1}(T(g)_-)_- = 0$. Also, letting $f = T(g)_-$ we see that $T^{-1}(T(g)_-)_+ = 0$. Therefore Equation 5.11 becomes $g = T^{-1}(T(g)_+)_+$ as required. □

Remark 5.3.2. Abstractly it is easy to see that A_+ is isomorphic to B_+ by noting that $A_+ \simeq A/A_-$ and $B_+ \simeq B/B_-$

As is Section 5.3.1, consider the torus $\mathbb{T}_\tau$ with puncture and local coordinate ϕ^{-1}. Let $\Sigma_\tau = \mathbb{T}_\tau \setminus \phi(\Delta)$ be the torus with boundary parametrization ϕ. Then $\gamma = \phi(S^1) = \partial\Sigma_\tau$ is an analytic on $\mathbb{T}_\tau$.

Remark 5.3.3. Note that ϕ determines a negative orientation of the boundary. Or in other words in this case we are considering a negatively oriented puncture. Everything in this section works equally well for a positively oriented boundary, but we stick with a particular orientation for ease of notation.

Let $\Omega^0(\gamma)$ be the space of smooth complex-valued functions on γ, and let

$$\Omega^0_J(\gamma) = \{g \in \Omega^0(\gamma) \,|\, g \text{ satisfies equation (5.10) }\}$$

This is the space of smooth functions for which the jump problem can be solved. This explains the notation, "J". Let $g \in \Omega^0_J(\gamma)$. The solution to the jump problem can be used to split g into the sum of two pieces g^+ and g^- where g^+ extends holomorphically outside γ and g^- extends holomorphically inside γ.

Remark 5.3.4. The sign convention is that $+$ will refer to functions that can be extended inside the Riemann surface. In our case, inside Σ_τ is the same as outside the curve γ.

We need to know that both the functions g^+ and g^- are smooth.

Lemma 5.3.5. *If* $g \in \Omega^0(\gamma)$ *then the analytic functions defined by*

$$F(q) = \int_\gamma g(p)M(p,q)dp$$

are smooth on γ.

Outline proof. In genus-zero the kernel $M(p,q)$ is just $d\zeta/(\zeta - z)$. In this case the boundary values of F are smooth. See Gakhov [19, Section 4.4] or Huang [30, Lemma D.4.2] for a proof of this classical result. As noted in Appendix B, the kernel $M(p,q)$ is constructed so that as $p \to q$

$$M(p,q)dp = \frac{dp}{p-q} + \text{regular terms.}$$

This is really the crucial property of the kernel as is pointed out in Zverovich [52, Section 2]. The singularity is the same as in the genus-zero case and so the proof of smoothness will carry over to the general case. ∎

Because the only analytic functions on the torus are constants, $g^{\pm}$ are unique up to the additon of constants. To see this, assume g has two decompositions $g = g_1^+ + g_1^- = g_2^+ + g_2^-$ and let $G_{1,2}^{\pm}$ be the corresponding analytic extensions. Because $(g_2^+ - g_1^+) + (g_2^- - g_1^-) = 0$ the boundary values of $G_2^+ - G_1^+$ and $-(G_2^- - G_1^-)$ agree and so we get a holomorphic function defined on the whole torus. This holomorphic function must a constant and so $g_2^+ - g_1^+ = -(g_2^- - g_1^-)$ is a constant.

Remark 5.3.6. Up to a constant $g^{\pm}$ is equal to $F^{\pm}$, which are the limiting values of F as the boundary is approached.

Using this decomposition, $\Omega_J^0(\gamma)$ can be written as a sum (not direct) of two vector subspaces

$$\Omega_{\text{in}}^0(\gamma) = \{g \in \Omega_J^0(\gamma) \,|\, g \text{ extends analytically inside } \Sigma_\tau\}$$

and

$$\Omega_{\text{out}}^0(\gamma) = \{g \in \Omega_J^0(\gamma) \,|\, g \text{ extends analytically inside } \gamma\}.$$

Let C_γ be the space of constant functions on γ. The existence and uniqueness (up to a constant) of the decomposition $g = g^+ + g^-$ ensures that

$$\Omega_J^0(\gamma) = \Omega_{\text{in}}^0(\gamma) + \Omega_{\text{out}}^0(\gamma)$$

and

$$\Omega_{\text{in}}^0(\gamma) \cap \Omega_{\text{out}}^0(\gamma) = C_\gamma$$

For later reference we record the following simple lemma.

Lemma 5.3.7. *The map* $\text{Hol}(\Sigma_\tau) \to \Omega_{in}^0(\gamma)$ *given by* $F \mapsto F|_\gamma$ *is an isomorphism.*

Proof. By definition, any $F \in \text{Hol}(\Sigma_\tau)$ is smooth up to the boundary and so $F|_\gamma \in \Omega_{J,\text{in}}^0(\gamma)$. □

Remark 5.3.8. It is worth pointing out again that the following discussion is only valid for the case when ϕ determines a negative orientation of the boundary.

Remark 5.3.9. Due mostly to notational difficulty, the work leading up to Corollary 5.3.15 on page 101 looks more complicated than it is.

Smooth functions on the circle can be split into their positive and negative Fourier components. Let

$$\Omega^0_{>0}(S^1) = \left\{ f \in \Omega^0(S^1) \,|\, f = \sum_{n=1}^{\infty} a_n \mathrm{e}^{in\theta} \right\},$$

$$\Omega^0_{<0}(S^1) = \left\{ f \in \Omega^0(S^1) \,|\, f = \sum_{n=-1}^{-\infty} a_n \mathrm{e}^{in\theta} \right\}$$

and

$$\Omega^0_0(S^1) = \{ f \in \Omega^0(S^1) \,|\, f = a_0 \}$$

The last space is just the space of constant functions. We have a direct sum decomposition

$$\Omega^0(S^1) = \Omega^0_{<0}(S^1) \oplus \Omega^0_0(S^1) \oplus \Omega^0_{>0}(S^1)$$

The isomorphism $\Omega^0(\gamma) \to \Omega^0(S^1)$ given by $g \mapsto g \circ \phi$ induces a splitting of $\Omega^0(\gamma)$ into

$$\Omega^0(\gamma) = \Omega^0_{<0}(\gamma) \oplus \Omega^0_0(\gamma) \oplus \Omega^0_{>0}(\gamma)$$

where

$$\Omega^0_{<0}(\gamma) = \{ g \in \Omega^0(\gamma) \,|\, g \circ \phi \in \Omega^0_{<0}(S^1)$$

and the other parts are defined analogously. Note that $\Omega^0_0(\gamma) = C_\gamma$ is the space of constant functions because $g \circ \phi$ is constant if and only if g is constant. As $\Omega^0_J(\gamma)$ is a subspace of $\Omega^0(\gamma)$ we get a splitting

$$\Omega^0_J(\gamma) = \Omega^0_{J,<0}(\gamma) \oplus \Omega^0_{J,0}(\gamma) \oplus \Omega^0_{J,>0}(\gamma) \tag{5.12}$$

where $\Omega^0_{J,<0}(\gamma) = \Omega^0_J(\gamma) \cap \Omega^0_{<0}(\gamma)$ and the other subspaces are defined in the analogous way. Given $f \in \Omega^0_J(\gamma)$ the component of f in $\Omega^0_{J,<0}(\gamma)$ will be written $f_{<0}$, and corresponding notation will be used for the other subspaces. As all the constant functions are in $\Omega^0_J(\gamma)$ we see that $\Omega^0_{J,0}(\gamma) = C_\gamma$.

We now want to relate these spaces to the subspaces $\Omega^0_{\mathrm{in}}(\gamma)$ and $\Omega^0_{\mathrm{out}}(\gamma)$. In particular we want to prove that $\Omega^0_{\mathrm{in}}(\gamma)$ is isomorphic to $\Omega^0_{J,<0}(\gamma) \oplus C_\gamma$. This will turn out to be useful in understanding the cokernel of π_Σ (see Section 5.3.3).

It is convenient to use the notation $\Omega^0_{J,\neq 0}(\gamma) = \Omega^0_{J,<0}(S^1) \oplus \Omega^0_{J,>0}(S^1)$. Let

$$\Omega^0_{\mathrm{in(out)},\neq 0}(\gamma) = \Omega^0_{\mathrm{in(out)}}(\gamma) \cap \Omega^0_{J,\neq 0}(\gamma)$$

Now we have a second direct sum decomposition of $\Omega^0_J(\gamma)$ given by

$$\Omega^0_J(\gamma) = \Omega^0_{\mathrm{in},\neq 0}(\gamma) \oplus C_\gamma \oplus \Omega^0_{\mathrm{out},\neq 0}(\gamma). \tag{5.13}$$

Lemma 5.3.10. *The space $\Omega^0_{out,\neq 0}(\gamma)$ is equal to $\Omega^0_{J,>0}(\gamma)$ and $\Omega^0_{in,\neq 0}(\gamma)$ is isomorphic to $\Omega^0_{J,<0}(\gamma)$. The isomorphism is given by the projection $f \mapsto f_{<0}$. (Equivalently, $\Omega^0_{out}(\gamma) = \Omega^0_{J,>0}(\gamma) \oplus C_\gamma$ and $\Omega^0_{in}(\gamma) \simeq \Omega^0_{J,<0}(\gamma) \oplus C_\gamma$.)*

Proof. If $g \in \Omega^0_{\mathrm{out},\neq 0}(\gamma)$ then by definition g extends analytically inside γ. As ϕ is a biholomorphism, $g \circ \phi$ extends analytically inside S^1 and thus $g \circ \phi \in \Omega^0_{>0}(S^1)$. Similarly, if $f \in \Omega^0_{>0}(S^1)$ then $f \circ \phi^{-1} \in \Omega^0_{\mathrm{out},\neq 0}(\gamma)$. Consider the identity map $I : \Omega^0_J(\gamma) \to \Omega^0_J(\gamma)$ and the diagram

$$\begin{array}{ccccccc}
\Omega^0_J(\gamma) & = & \Omega^0_{\mathrm{in},\neq 0} & \oplus & C_\gamma & \oplus & \Omega^0_{\mathrm{out},\neq 0} \\
\downarrow I & & \downarrow & & \downarrow I & & \downarrow I \\
\Omega^0_J(\gamma) & = & \Omega^0_{J,<0}(\gamma) & \oplus & \Omega^0_{J,0}(\gamma) & \oplus & \Omega^0_{J,>0}(\gamma).
\end{array}$$

We have just showed that $I(\Omega^0_{\mathrm{out},\neq 0}(\gamma)) = \Omega^0_{J,>0}(\gamma)$. Also we noted above that $\Omega^0_{J,0}(\gamma) = C_\gamma$. Now Lemma 5.3.1 can be applied with $T = I$ and therefore $\Omega^0_{\mathrm{in},\neq 0}(\gamma)$ is isomorphic to $\Omega^0_{J,<0}(\gamma)$. The isomorphism is just the projection map onto $\Omega^0_{J,<0}(\gamma)$. Since $\Omega^0_{\mathrm{in}}(\gamma) = \Omega^0_{\mathrm{in},\neq 0}(\gamma) \oplus C_\gamma$ we also see that $\Omega^0_{\mathrm{in}}(\gamma)$ is isomorphic to $\Omega^0_{J,>0}(\gamma) \oplus C_\gamma$. $\square$

We need to further investigate equation (5.10), in order to understand the space $\Omega^0_J(\gamma)$. By considering γ as parametrized by $p = \phi(\exp(i\theta))$, equation (5.10) becomes, after a little work,

$$\int_0^{2\pi} (g \circ \phi)(\mathrm{e}^{i\theta}) i \mathrm{e}^{i\theta} \phi'(\mathrm{e}^{i\theta}) d\theta = 0. \tag{5.14}$$

This condition can now be understood as saying that the negative-first Fourier coefficient of $(g \circ \phi)(\mathrm{e}^{i\theta})\phi'(\mathrm{e}^{i\theta})$ must equal zero. It follows that $\Omega^0_J(\gamma)$ is a codimension one subspace of $\Omega^0(\gamma)$. This will be discussed further in Section 5.3.3.

Remark 5.3.11. Although not explicitly used, the space of functions on the circle

$$\Omega^0_\phi(S^1) = \{f \in \Omega^0(S^1) \,|\, f \circ \phi^{-1} \in \Omega^0_J(\gamma)\}$$

may be conceptually helpful. There is an isomorphism between $\Omega^0_J(\gamma)$ and $\Omega^0_\phi(S^1)$ given by $g \mapsto (g \circ \phi)$. The reason this space is not useful is that it depends on ϕ.

To produce a trivialization we need a space that does not depend on ϕ. In light of equation (5.14) we define the space

$$\Omega^0_{\neq -1}(S^1) = \left\{ f \in \Omega^0(S^1) \,|\, \int_0^{2\pi} f \mathrm{e}^{i\theta} d\theta = 0 \right\}.$$

Note that this is just the space of smooth function on the circle with vanishing negative-first Fourier coefficient.

Consider the operator $T : \Omega^0(\gamma) \longrightarrow \Omega^0(S^1)$ defined by

$$T(g) = (g \circ \phi)(\mathrm{e}^{i\theta})\phi'(\mathrm{e}^{i\theta}). \tag{5.15}$$

Lemma 5.3.12. *The map T defined by equation (5.15) is an isomorphism.*

Proof. Because ϕ is a biholomorphism, $\phi'(z) \neq 0$ for all z. So for any $f \in \Omega^0(S^1)$,

$$T^{-1}(f) = \left(\frac{f}{\phi'}\right) \circ \phi^{-1}$$

is in $\Omega^0(\gamma)$. □

Lemma 5.3.13. *The map T restricts to an isomorphism*

$$T : \Omega^0_J(\gamma) \longrightarrow \Omega^0_{\neq -1}(S^1).$$

Proof. From Equation 5.14 we see that $T(g)$ satisfies

$$\int_0^{2\pi} T(g)\mathrm{e}^{i\theta} d\theta = 0$$

and so is in $\Omega^0_{\neq -1}(S^1)$. □

In keeping with our notation we define

$$\Omega^0_{<-1}(S^1) = \left\{ f \in \Omega^0(S^1) \,|\, f = \sum_{n=-2}^{-\infty} a_n \mathrm{e}^{in\theta} \right\}$$

and note that there is a direct sum decomposition

$$\Omega^0_{\neq -1}(S^1) = \Omega^0_{<-1}(S^1) \oplus \Omega^0_0(S^1) \oplus \Omega^0_{>0}(S^1).$$

For $f \in \Omega^0(S^1)$, let $f_{<-1}$ be the components in $\Omega^0_{<-1}(S^1)$. Similar notation will be employed for the other subspaces.

Proposition 5.3.14. *The space of function $\Omega^0_{J,<0}(\gamma)$ is isomorphic to $\Omega^0_{<-1}(S^1)$.*

Proof. The idea is to apply Lemma 5.3.1 to the isomorphism $T : \Omega^0_J(\gamma) \to \Omega^0_{\neq -1}(S^1)$ of Lemma 5.3.13 and the decompositions

$$\begin{array}{ccccc} \Omega^0_J(\gamma) & = & \Omega^0_{J,<0}(\gamma) & \oplus & \big(C_\gamma \quad \oplus \ \Omega^0_{J,>0}(\gamma)\big) \\ {\scriptstyle T}\big\downarrow & & & & \\ \Omega^0_{\neq -1}(S^1) & = & \Omega^0_{<-1}(S^1) & \oplus & \big(\Omega^0_0(S^1) \ \oplus \ \Omega^0_{>0}(S^1)\big). \end{array}$$

To satisfy the hypothesis of Lemma 5.3.1 we need to show that $T(C_\gamma \oplus \Omega^0_{J,>0}(\gamma)) = \Omega^0_{>0}(S^1) \oplus \Omega^0_0(S^1)$.

First note that $T(C_\gamma) = \Omega^0_0(S^1)$. If $g \in \Omega^0_{J,>0}(\gamma)$ then g extends to a holomorphic function inside γ. Because ϕ is a biholomorphism, $T(g)$ extends to a holomorphic function inside S^1. Similarly, if $f \in \Omega^0_{>0}(S^1)$ then f extends to a holomorphic function inside S^1. As noted earlier $\phi'(z) \neq 0$ and so $T^{-1}(f) = (f/\phi') \circ \phi^{-1}$ extends to a holomorphic function inside γ.

So Lemma 5.3.1 applies and the isomorphism $\Omega^0_{J,<0}(\gamma) \to \Omega^0_{<-1}(S^1)$ is given by the map $f \mapsto T(f)_{<-1}$. □

Combining Lemmata 5.3.7 and 5.3.10 with Proposition 5.3.14 results in the important trivialization of the space of holomorphic functions on the torus with parametrized boundary. Recall that $\Sigma_\tau = \mathbb{T}_\tau \setminus \phi(\Delta)$ is a genus one Riemann surface with boundary $\gamma = \phi(S^1)$.

Corollary 5.3.15. *The spaces* $\mathrm{Hol}(\Sigma_\tau)$ *and* $\Omega^0_{<-1}(S^1) \oplus \Omega^0_0(S^1)$ *are isomorphic. The isomorphism is given by the sequence*

$$\mathrm{Hol}(\Sigma_\tau) \longrightarrow \Omega^0_{in}(\gamma) \longrightarrow \Omega^0_{J,<0}(\gamma) \oplus C_\gamma \longrightarrow \Omega^0_{<-1}(S^1) \oplus \Omega^0_0(S^1),$$

where the maps are defined by

$$F \longmapsto f = F|_\gamma \longmapsto f_{<0} \longmapsto (T(f_{<0}))_{<-1}\,.$$

Remark 5.3.16. The above isomorphism could be achieved without using the intermediate space $\Omega^0_{J,<0}(\gamma)$. The reason for introducing these spaces is so that the cokernel of the map $\pi_{\Sigma_\tau} : \mathrm{Hol}(\Sigma_\tau) \to \Omega^0_+(\gamma) = \Omega^0_{J,<0}(\gamma)$ can be understood. See Section 5.3.3.

We now prove that the spaces $\mathrm{Hol}(\Sigma_\tau)$ form an infinite-dimensional fiber space over the Teichmüller space $\widetilde{T}_B(\Sigma)$. The above isomorphism gives a trivialization and thus a bundle structure.

Theorem 5.3.17. *For any genus-one surface Σ with one puncture, the space*

$$\bigsqcup_{[\Sigma,f,\Sigma_1,\phi]\in\widetilde{T}_B(\Sigma)} \mathrm{Hol}(\Sigma_\tau)$$

is a holomorphic bundle over the Teichmüller space $\widetilde{T}_B(\Sigma)$.

Proof. The only thing to worry about is that $\mathrm{Hol}(\Sigma_\tau)$ and its trivialization are well defined. Any equivalence class $[\Sigma, f, \Sigma_1, \psi]$ has a unique representative of the form $[\Sigma, g, \mathbb{T}_\tau, \phi]$, where ϕ is related to ψ by the unique biholomorphism $\sigma : \Sigma_1 \to \mathbb{T}_\tau$. Section 4.6 in Nag [44] contains a detailed discussion of the Teichmüller space of the once-punctured torus.

So ϕ and Σ_τ are uniquely determined by the equivalence class $[\Sigma, f, \Sigma_1, \psi]$. Therefore $\mathrm{Hol}(\Sigma_\tau)$ and its trivialization, which is determined by ϕ, are well defined. ☐

5.3.3 Cokernel

Recall from Section 5.1 that the determinant line is formed from the top exterior powers of the kernel and cokernel of

$$\pi_\Sigma : \mathrm{Hol}(\Sigma) \longrightarrow \Omega^0_+(\partial\Sigma).$$

The kernel is either zero or the space of constant functions (see Huang [30]). In the genus-zero case the cokernel is always zero. In genus-one the cokernel is non-trivial because of the obstruction to the solution of the jump problem. We will produce a trivialization of the cokernel which in turn will lead to a bundle structure for the determinant lines.

Recall from Section 5.3.2 that $\Sigma_\tau = \mathbb{T}_\tau \setminus \phi(\Delta)$ where $\mathbb{T}_\tau$ is the parallelogram $0, \tau, 1, 1+\tau$ with opposite edges identified. Also, let $\gamma = \partial\Sigma_\tau = \phi(S^1)$ as before. In Lemma 5.3.12 it was shown that $T : \Omega^0(\gamma) \to \Omega^0(S^1)$, defined by $T(g) = (g \circ \phi)\phi'$, is an isomorphism.

We consider only the case when ϕ determines a negative orientation of the boundary component (the other case is similar). This means that $\phi : \Delta \to \mathbb{T}_\tau$ is an orientation preserving biholomorphism. In this case $\Omega^0_+(\partial\Sigma_\tau) = \Omega^0_{<0}(\gamma)$.

Let $\Omega^0_{-1}(S^1) = \{f \in \Omega^0(S^1) \,|\, f = a_{-1}\mathrm{e}^{-in\theta}\}$. For $f \in \Omega^0(S^1)$ the component of f in $\Omega^0_{-1}(S^1)$ is written f_{-1}, and the component in $\Omega^0_{<0}(S^1)$ is written $f_{<0}$.

Lemma 5.3.18. *The cokernel of π_{Σ_τ} is isomorphic to $\Omega^0_{-1}(S^1)$ and moreover, the isomorphism is induced by the map T.*

Proof. This proof is actually straightforward given our previous results. It is just the notation which is troublesome. The basic idea is that π_{Σ_τ} is not surjective because there are functions in $\Omega^0_+(\Sigma_\tau)$ which cannot be extended to Σ_τ, and these functions are the ones that do not satisfy the jump condition. We already showed that $\Omega^0_{J,<0}$ is isomorphism to $\Omega^0_{<-1}(S^1)$. So it is just the one-dimensional space $\Omega^0_{-1}(S^1)$ which is missed.

Recall that $\Omega^0_{J,<0}(\gamma) \oplus C_\gamma \subset \Omega^0_{\leq 0}(\gamma)$ is isomorphic to both $\Omega^0_{<-1}(S^1) \oplus \Omega^0_0(S^1)$ and $\mathrm{Hol}(\Sigma_\tau)$. Given $g \in \Omega^0_+(\partial\Sigma_\tau) = \Omega^0_{<0}(\gamma)$, we see that $g \notin \pi_{\Sigma_\tau}(\mathrm{Hol}(\Sigma_\tau))$ if and only if

$$\int_\gamma g(p)dp \neq 0.$$

Let $\mathcal{M} = T^{-1}(\Omega^0_{-1}(S^1))$. Then the direct sum decomposition $\Omega^0(S^1) = \Omega^0_{\neq -1}(S^1) \oplus \Omega^0_{-1}(S^1)$ induces a decomposition

$$\Omega^0(\gamma) = \Omega^0_J(\gamma) \oplus \mathcal{M}.$$

Let $\mathcal{M}_{<0}$ be the projection of $\mathcal{M}$ onto $\Omega^0_{<0}(\gamma)$. We want to show that $T(\mathcal{M}_{<0})_{<0} =$

$\Omega^0_{-1}(S^1)$. If $g \in \mathcal{M}$, then

$$(g \circ \phi)\phi' = b_{-1}\mathrm{e}^{-i\theta}$$

which implies

$$g \circ \phi = \frac{b_{-1}\mathrm{e}^{-i\theta}}{\phi'}.$$

Because ϕ is a biholomorphism, ϕ' is never zero and so can be expanded as

$$\frac{1}{\phi'} = a_0 + \sum_{n>0} a_n \mathrm{e}^{in\theta}$$

where $a_0 \neq 0$. Therefore,

$$g \circ \phi = a_0 b_{-1}\mathrm{e}^{-i\theta} + b_{-1}\sum_{n>0} a_i \mathrm{e}^{i(n-1)\theta}$$

and the component of $g \circ \phi$ in $\Omega^0_{<0}(S^1)$ is $a_0 b_{-1} \exp^{-i\theta}$. So g is in $\mathcal{M}_{<0}$ if and only if $g \circ \phi = a_0 b_{-1} \exp^{-i\theta}$. Let $g \in \mathcal{M}_{<0}$ and let the expansion of ϕ' be c_0+higher order terms. By definition of T, we get

$$T(g)_{<0} = \big((g \circ \phi)\phi'\big)_{<0} = c_0 a_0 b_{-1}\mathrm{e}^{-i\theta}.$$

Because a_0 and c_0 are non-zero and b_{-1} is arbitrary, $T(\mathcal{M}_{<0})_{<0} = \Omega^0_{-1}(S^1)$.

Consider the decomposition

$$\Omega^0_{<0}(\gamma) = \Omega^0_{J,<0}(\gamma) \oplus \mathcal{M}. \tag{5.16}$$

It follows from Lemma 5.3.7 and the definition of π_Σ that $\pi_{\Sigma_\tau}(\mathrm{Hol}(\Sigma_\tau)) = \Omega^0_{J,+}(\gamma)$. So

$$\mathrm{Coker}(\pi_{\Sigma_\tau}) = \Omega^0_{<0}(\gamma)/\Omega^0_{J,<0}(\gamma),$$

which is isomorphic to $\mathcal{M}$ by equation (5.16). We showed above that $T(\mathcal{M})_{<0} = \Omega^0_{-1}(S^1)$, so the cokernel is isomorphic to the one-dimensional space $\Omega^0_{-1}(S^1)$. Recapping, we have

$$\mathrm{Coker}(\pi_{\Sigma_\tau}) = \Omega^0_{<0}(\gamma)/\Omega^0_{J,<0}(\gamma) \simeq \mathcal{M} \xrightarrow{T_{<0}} \Omega^0_{-1}(S^1)$$

which is the desired isomorphism. □

Remark 5.3.19. Perhaps a better conceptual way to view the trivialization of the cokernel is to consider the isomorphism

$$\Omega^0_{<0}(\gamma)/\Omega^0_{J,<0}(\gamma) \longrightarrow \Omega^0_{<0}(S^1)/\Omega^0_{<-1}(S^1)$$

given by

$$[g] \longmapsto [T(g)_{<-1}].$$

Surjectivity follows form above and injectivity follows from the equivalences

$$\begin{aligned} [g] \neq 0 &\Longleftrightarrow \int_\gamma g(p)dp \neq 0 \\ &\Longleftrightarrow \int_{S^1} T(g)ie^{i\theta} \neq 0 \\ &\Longleftrightarrow T(g)_{-1} \neq 0 \\ &\Longleftrightarrow [T(g)_{<0}] \neq 0. \end{aligned}$$

Using the trivialization of the cokernel in Lemma 5.3.18 we get a holomorphic bundle just as in Theorem 5.3.17.

Corollary 5.3.20. *For any genus one surface Σ with one boundary component the space*

$$\bigsqcup_{[\Sigma,f,\Sigma_1,\phi]\in\widetilde{T}_B(\Sigma)} \operatorname{Coker}(\pi_{\Sigma_\tau})$$

is a holomorphic bundle over the Teichmüller space $\widetilde{T}_B(\Sigma)$.

5.3.4 Determinant line bundle

Let Σ be any genus-one surface with one puncture. As before, let $\Sigma_\tau = \mathbb{T}_\tau \setminus \phi(\Delta)$. Given $[\Sigma, g, \Sigma_1, \psi] \in \widetilde{T}_B(\Sigma)$, there are representatives of the form $(\Sigma, f, \mathbb{T}_\tau, \phi^{-1})$, where ϕ^{-1} is related to ψ by the biholomorphism $\sigma : \Sigma_1 \to \mathbb{T}_\tau$. The fact that f is not unique does not effect the uniqueness of $\operatorname{Det}_{\mathbb{T}_\tau}$.

The *determinant line associated to a Teichmüller space element* $[\Sigma, g, \Sigma_1, \psi]$ is defined to be

$$\operatorname{Det}_{[\Sigma,g,\Sigma,\psi]} = \operatorname{Det}_{\Sigma_\tau},$$

where τ is such that $[\Sigma, g, \Sigma, \psi] = [\Sigma, f, \mathbb{T}_\tau, \phi^{-1}]$. The boundary parametrization of Σ_τ is ϕ^{-1}. The fiber space of determinant lines over $\widetilde{T}_B(\Sigma)$ is

$$\mathrm{Det}_{\widetilde{T}_B(\Sigma)} = \bigsqcup_{[\Sigma, f, \Sigma_\tau, \psi] \in \widetilde{T}_B(\Sigma)} \mathrm{Det}_{\Sigma_\tau}$$

Theorem 5.3.21. *If Σ is any genus-one surface with one puncture then the determinant lines form a trivial holomorphic line bundle over the Teichmüller space $\widetilde{T}_B(\Sigma)$. That is, the fiber space $\mathrm{Det}_{\widetilde{T}_B(\Sigma)}$ is a trivial holomorphic line bundle.*

Proof. The determinant line of Σ_τ is constructed from $\mathrm{Ker}(\pi_{\Sigma_\tau})$ and $\mathrm{Coker}(\pi_{\Sigma_\tau})$ using the operations of forming exterior products, tensor products and taking the dual. These are standard operations in vector bundle theory. So it is sufficient to show that $\mathrm{Ker}(\pi_{\Sigma_\tau})$ and $\mathrm{Coker}(\pi_{\Sigma_\tau})$ form holomorphic vector bundles.

In any genus we know, by Huang[30, Corollary D.3.], that the kernel is either zero or the space of constant functions. The bundle structure of the cokernel follows from Lemma 5.3.18 and Corollary 5.3.20. Since $\Omega^0_{-1}(S^1)$ is isomorphic to $\mathbb{C}$, it follows that for any Σ_τ we have an explicit isomorphism $\mathrm{Coker}(\Sigma) \to \mathbb{C}$. We conclude that $\mathrm{Ker}(\pi_{\Sigma_\tau})$, $\mathrm{Coker}(\pi_{\Sigma_\tau})$, and thus $\mathrm{Det}_{\widetilde{T}_B(\Sigma)}$, form trivial holomorphic vector bundles over $\widetilde{T}_B(\Sigma_\tau)$. □

Remark 5.3.22. There is another method of proof which does not explicitly reply on understanding the cokernel. We have shown that the spaces $\mathrm{Hol}(\Sigma_\tau)$ form a holomorphic vector bundle over the rigged Teichmüller space. Part of the proof already contains the fact that $\Omega^0_+(\partial\Sigma_\tau)$ also forms a holomorphic vector bundle and moreover, that π_{Σ_τ} is a holomorphic vector bundle map. It is a standard result (see Husemoller [35]) that if such a map is of constant rank then its kernel and cokernel are holomorphic vector bundles.

5.4 Holomorphicity of genus-zero self-sewing

Our aim is to prove the holomorphicity of the self-sewing isomorphism for genus-zero surfaces. Note that the resultant surface has genus equal to one.

In Section 5.2 the construction of the sewing isomorphism was carried out for arbitrary genus. The main new thing that had to be proved was the surjectivity of the Δ map. This was done in Section 5.2.3. We briefly repeat the proof for our current special case.

We only need to consider a genus-one surface $\hat{\Sigma}$ with one puncture and local coordinate, say ϕ. Let $D = \phi^{-1}(\Delta)$. Cutting out D we get surface $\Sigma = \hat{\Sigma} \setminus D$ with one boundary component, say S_3. Let S be a simple closed curve so that cutting Σ along S produces a sphere with three boundary components.

In general we know the jump problem cannot be solved. The obstruction

$$\int_S g(p) dZ_p = 0$$

is the vanishing of the singular part of

$$F = \int_S g(q) M(p,q) dZ_q$$

at the pole q_1. (See Appendix B or Rodin [47, page 25] for details.)

Remark 5.4.1. As previously noted the fact that S does not separate the surface can be fixed by adding extra curves along which the jump is zero.

For the torus the construction of the Cauchy-kernel $M(p,q)$ such that F has is single valued requires the addition of an extra pole at arbitrary point q_1. By choosing $q_1 \in D$ we see that F is actually holomorphic on Σ and so the jump problem can be solved for any g.

Let $\Sigma_{1,2,3}$ be a genus-zero Riemann surface with 3 boundary components for which boundaries S_1 and S_2 can be sewn. Note that the resulting surface is a torus with one boundary components. For any genus-zero surface $\Sigma'_{1,2,3}$ we have the sewing isomorphism $\ell_{\Sigma'_{1,2,3};S_1,S_2} : \mathrm{Det}_{\Sigma'_{1,2,3}} \to \mathrm{Det}_{\Sigma'_{1\#2,3}}$.

Theorem 5.4.2. *Self-sewing of genus-zero surfaces with three boundary components is holomorphic. That is, the sewing map*

$$\begin{aligned} \mathcal{L}_{\Sigma_{1,2,3;S_1,S_2}} : \mathrm{Det}_{\widetilde{T}_B(\Sigma_{1,2,3})} &\longrightarrow \mathrm{Det}_{\widetilde{T}_B(\Sigma_{1\#2,3})} \\ \left([\Sigma'_{1,2,3}, \phi_1, \phi_2, \phi_3], \lambda\right) &\longmapsto \left([\Sigma'_{1\#2,3}, \phi_3], \ell_{\Sigma'_{1,2,3};S_1,S_2}(\lambda)\right) \end{aligned}$$

is a holomorphic map of vector bundles

Outline of proof. All the steps in Huang [30] of the proof of the holomorphicity of sewing can now be emulated for genus-zero to genus-one self-sewing. In particular we have shown that $\mathrm{Hol}(\Sigma_\tau)$, $\mathrm{Ker}(\pi_\tau)$ and $\mathrm{Coker}(\pi_\tau)$ form holomorphic bundles. We have also shown that the construction of the sewing isomorphism can be carried over to arbitrary genus surfaces. □

5.5 Pants decompositions

5.5.1 A result of Hatcher and Thurston

This section is a precis of some of the results and terminology developed in the work of Hatcher and Thurston [24], Hatcher, Lochak and Schneps [22] and Hatcher [23]. See also Luo [40] for related work.

Surfaces of genus-zero with three boundary components are called *pants*. Any surface can be decomposed into pants by cutting along sufficiently many closed curves. An example of a pants decomposition is illustrated in Figure 5.1 which shows a surface of type $(1, 0, 3)$ being decomposed into three pairs of pants. Because we need to study arbitrary pants decompositions of a surface, the relationship between different decompositions is important. Pioneering work on this relationship was done by Hatcher and Thurston in [24]. An explicit discussion of pants decompositions only appears in the appendix of their work. More details and further results appear in Hatcher, Lochak, and Schneps [22]. The part of their paper related to pants decompositions can be found as an independent preprint by Hatcher [23]. We closely follow the notation and language of the later paper [22] (or equivalently [23]).

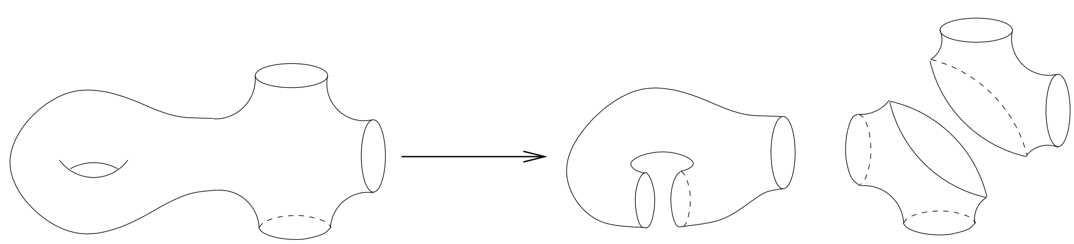


Figure 5.1: Pants decomposition

Recall that surfaces of genus g with n punctures and m boundary components are said to be of topological type (g, n, m). In this language, pants are surfaces of type $(0, 0, 3)$. The term 3-holed sphere is sometimes used.

Let Σ be a Riemann surface of type $(g, 0, m)$. A *maximal multicurve* on Σ is a finite collection $\mathcal{P}$ of disjoint smoothly embedded circles such that cutting along these curves decomposes Σ into pants. We say that $\mathcal{P}$ defines a *pants decomposition* of Σ. The number of curves in $\mathcal{P}$ is $3g - 3 + m$ and the number of pairs of pants in the decomposition is $2g - 2 + m$.

One of the fundamental results of Hatcher and Thurston [24] is that any two isotopy classes of pants decompositions can be joined by a sequence *elementary moves* which we describe below.

Remark 5.5.1. Hatcher and Thurston further prove that the two-dimensional cell complex whose vertices are pants decompositions and edges are elementary moves is simply connected. We do not need this stronger result.

Remark 5.5.2. Later we will consider specific pants decompositions and not just isotopy classes.

We proceed by describing the two elementary moves. See Figures 5.2 and 5.3.

S-move: Suppose that there is a curve a in $\mathcal{P}$ such that after removing a, one of the components of the decomposition is of type $(1, 0, 1)$. Then there is a curve b in Σ that intersects a transversally at one point and is disjoint from all other curves in $\mathcal{P}$. Replacing a by b produces a new pants decomposition $\mathcal{P}'$. We call such a replacement an *S-move.* (See Figure 5.2.)

A-move: Suppose that there is a curve a in $\mathcal{P}$ such that after removing a, one of the components of the decomposition is of type $(0, 0, 4)$. Then there is a curve b in Σ that intersects a transversally at two points and is disjoint from all other curves in $\mathcal{P}$. Replacing a by b produces a new pants decomposition $\mathcal{P}'$. We call such a replacement an *A-move.*(See Figure 5.3.)

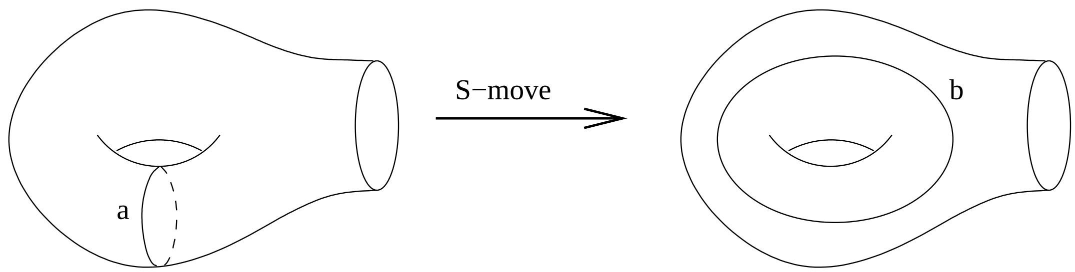


Figure 5.2: S-move

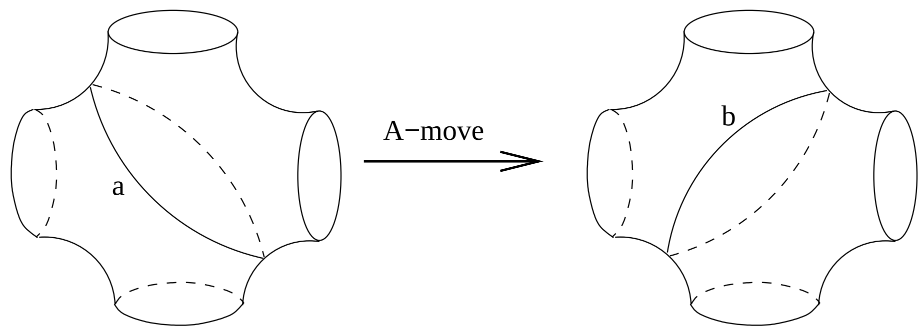


Figure 5.3: A-move

For future reference we state part of the result of Hatcher and Thurston in following theorem.

Theorem 5.5.3. *Any two isotopy classes of pants decompositions can be joined by a finite sequence of A-moves and S-moves.*

5.5.2 Homotopy move

We need to work with particular pants decompositions rather than isotopy classes of these decompositions. At first an extra operation will be introduced in order to relate pants decompositions where the curves may be homotopically equivalent. We then show this extra operation can actually be obtained from the elementary moves.

Let $\mathcal{P}$ be a pants decomposition of a surface Σ of type $(g, 0, m)$ and let $N = 3g - 3 + m$. Let a be a curve in $\mathcal{P}$. Assume b is homotopic to a and b does not intersect any of the curves in $\mathcal{P}$, except possibly a. Then replacing a with b defines a new pants decomposition. We call such a replacement an *H-move*, where "H" stands for "homotopy.

Remark 5.5.4. There is no need to consider H-moves explicitly as they can be absorbed into the S- and A-moves. In fact any H-move can be expressed as either two A-moves on a sphere with four boundaries or two as S-moves on a torus with one boundary.

Let $\mathcal{P} = (S_1, \dots S_N)$ and $\mathcal{P}' = (S'_1, \dots S'_N)$ be pants decompositions of Σ such that S_i is homotopic to S'_i, for $i = 1, \dots, N$. The following lemma is a straightforward exercise with the use of the appropriate topological facts.

Remark 5.5.5. I would like to thank Feng Luo for showing me the proof of the following lemma and directing me to the appropriate literature.

Lemma 5.5.6. *The pants decompositions $\mathcal{P}$ and $\mathcal{P}'$ can be related by a sequence of H-moves.*

Proof. We may not be able to simply replace S_1 by S'_1 and so forth. This is because S'_1 may intersect some $S \in \mathcal{P}$ and in this case the replacement does produce a valid pants decomposition.

To use a result of Casson [10] we introduce some terminology. We specialize his definitions and results to our situation. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be pants decompositions of Σ. The *geometric intersection number* $i(\mathcal{P}_1, \mathcal{P}_2)$ is defined to be the minimum value of $|\mathcal{P}'_1 \cap \mathcal{P}'_2|$ where $\mathcal{P}'_1$ and $\mathcal{P}'_2$ are homotopic to $\mathcal{P}_1$ and $\mathcal{P}_2$ respectively. We say that $\mathcal{P}_1$ and $\mathcal{P}_2$ intersect minimally if $|\mathcal{P}_1 \cap \mathcal{P}_2| = i(\mathcal{P}_1, \mathcal{P}_2)$.

Casson [10, Lemma 2.5, page 26] proves that if $\mathcal{P}_1$ and $\mathcal{P}_2$ are pants decomposition of Σ, then $\mathcal{P}_2$ is isotopic to a pants decomposition having minimal intersection with $\mathcal{P}_1$. The method of proof is to apply "Whitney's trick" successively to reduce the intersection number. It is important to note that each of these operations involves only H-moves.

Returning to our situation we first note that $i(\mathcal{P}, \mathcal{P}') = 0$. To see this just move S'_i, by isotopy, to a sufficiently small neighborhood of S_i that does not contain any S_j, for all $j \neq i$. Applying Casson's result $\mathcal{P}$ and $\mathcal{P}'$ gives a pants decomposition $\mathcal{P}'' = (S''_1, \dots S''_N)$ such that $|\mathcal{P}'' \cap \mathcal{P}| = 0$.

Joining $\mathcal{P}''$ to $\mathcal{P}$ by a sequence of H-moves is simple as we can just successively replace S_i with S_i'', for $i = 1, \ldots, N$. Putting these two steps together joins $\mathcal{P}$ to $\mathcal{P}'$ by a sequence of H-moves. □

By applying Lemma 5.5.6 a slight strengthening of Theorem 5.5.3 of Hatcher and Thurston is obtained.

Corollary 5.5.7. *Any two arbitrary pants decompositions, $\mathcal{P}$ and $\mathcal{P}'$, can be joined by a finite sequence of S-moves and A-moves.*

Proof. By Remark 5.5.4 the use of H-moves is allowed as they can be produced from the other two moves. By applying Theorem 5.5.3 we can convert $\mathcal{P}$ into a pants decomposition that is in the same isotopy class as $\mathcal{P}'$. Lemma 5.5.6 can now be applied to obtain identical decompositions by using H-moves. □

Note that the sequence of S- and A-moves needed to obtain a relation between pants decompositions will in general be longer than the sequence needed to relate their isotopy classes.

5.6 Holomorphic bundle structure via pants decompositions

We first give a definition of the determinant line associated to a Teichmüller space element. This has already be done in the torus case and the idea is the same here.

Each equivalence class in Tecihmüller has canonical representatives of the form $[\Sigma, f^\mu, \Sigma^\mu, \phi]$, where $\phi = (\phi_1, \ldots, \phi_n)$. As discussed in Appendix A and Chapter 3, the Riemann surface Σ^μ only depends on the equivalence class $[\mu] \in T(G)$. However the map f^μ may depend on μ, and not just its equivalence class.

The determinant of the surface $\Sigma^\mu \backslash \phi^{-1}(\Delta)$ with boundary components parametrized by ϕ^{-1} does not depend on the choice of f^μ. Thus we have a unique assignment of a determinant line to a Teichmüller space element.

The goal of this Chapter, as well as of the thesis as a whole, is to prove the following.

Theorem 5.6.1. *The fiber space*

$$\mathrm{Det}_{\widetilde{T}_B(\Sigma)} = \bigsqcup_{[\Sigma, f, \Sigma_1, \phi_1, \ldots, \phi_n] \in \widetilde{T}_B(\Sigma)} \mathrm{Det}_{\Sigma^\mu f}$$

over the rigged Teichmüller space is a holomorphic line bundle.

Proof. First we give a rough outline of the ideas. Given a pants decomposition of a surface we get canonical isomorphisms

$$\mathrm{Det}_\Sigma \to \mathrm{Det}_{\Sigma_1} \otimes \ldots \otimes \mathrm{Det}_{\Sigma_N} \to \mathbb{C},$$

where the Σ_i are pants. To produce a bundle structure we need local trivializations. The problem is how to consistently choose pants decompositions on neighboring surfaces in Teichmüller space. Schiffer variation produces a neighborhood consisting of surfaces Σ^{μ_ϵ}. Recall that μ_ϵ is only non-zero on the disk $D \subset \Sigma$. So $f^{\mu_\epsilon} : \Sigma \to \Sigma^{\mu_\epsilon}$ is holomorphic on $\Sigma \backslash D$. If we choose a pants decomposition of Σ such that none of the curves intersect D, then f^{μ_ϵ} induces a pants decomposition of Σ^{μ_ϵ}. The crucial point is that an analytic curve on Σ is mapped by f^{μ_ϵ} to an analytic curve on Σ^{μ_ϵ}. Note that for general f this is not true because f is only quasiconformal.

Given two such trivializations we must show that the transition function is holomorphic. By the small extension to the results of Hatcher and Thurston in Corollary 5.5.7, we know that any two pants decompositions are related by a sequence of two types of basic moves. These two moves involve only genus-zero and genus-one surfaces. By using the associativity of the sewing isomorphism, the holomorphicity of the transition function can be reduced to the holomorphicity of the elementary moves. The genus-zero result is already known (see Huang [30]). The genus-zero to genus-one self-sewing has been proved in Section 5.4.

We begin by considering two pants decompositions for which only the parametrizations of the boundary components differ. It is sufficient to consider the case where such a change is only made on one curve in the decomposition. Let Σ a Riemann surface and consider a pants decomposition which includes a curve S that separates the two genus-zero pieces Σ_1 and Σ_2. These surfaces with different parametrizations will be called Σ_i, ϕ_i and Σ_i, ψ_i, for $i = 1, 2$. The two trivializations of Det_Σ are

$$\mathrm{Det}_\Sigma \longrightarrow \cdots \otimes \mathrm{Det}_{\Sigma_1,\phi_1} \otimes \mathrm{Det}_{\Sigma_2,\phi_2} \otimes \cdots \longrightarrow \mathbb{C}$$

and

$$\mathrm{Det}_\Sigma \longrightarrow \cdots \otimes \mathrm{Det}_{\Sigma_1,\psi_1} \otimes \mathrm{Det}_{\Sigma_2,\psi_2} \otimes \cdots \longrightarrow \mathbb{C}.$$

where the "$\cdots\otimes$" represents the other terms in the pants decomposition. To relate the transition function to genus-zero sewing operations, we consider a further cutting of the surface. Let C_1 and C_2 be curves on Σ_1 and Σ_2 respectively that are homotopic to, and disjoint from, S. Let A^1 and A^2 be the annuli formed by cutting along S, C_1 and C_2. To account for the different parametrizations we will write $A^1_{\phi_1}$ for the annulus A^1 with boundary parametrization ϕ along the curve corresponding to S. Corresponding notation will be used for the other cases. Let $A = A^1 \# A^2$ and let $\Sigma_1^0 = \Sigma_1 \setminus A^1$ and $\Sigma_2^0 = \Sigma_2 \setminus A^2$.

Consider the following diagram constructed using the sewing operation. (The terms indicated by "$\cdots\otimes$" above are dropped for simplicity.)

$$\begin{array}{ccc}
\mathrm{Det}_\Sigma & \longrightarrow & \mathrm{Det}_{\Sigma_1,\psi_1} \otimes \mathrm{Det}_{\Sigma_2,\psi_2} \\
\big\downarrow & \swarrow{\scriptstyle \ell_P} & \big\downarrow {\scriptstyle \ell^{-1}\otimes\ell^{-1}} \\
\mathrm{Det}_{\Sigma_1,\phi_1} \otimes \mathrm{Det}_{\Sigma_2,\phi_2} & & \mathrm{Det}_{\Sigma_1^0} \otimes \mathrm{Det}_{A^1_{\psi_1}} \otimes \mathrm{Det}_{A^2_{\psi_2}} \otimes \mathrm{Det}_{\Sigma_2^0} \\
{\scriptstyle \ell^{-1}\otimes\ell^{-1}} \big\downarrow & & \big\downarrow {\scriptstyle I\otimes\ell\otimes I} \\
\mathrm{Det}_{\Sigma_1^0} \otimes \mathrm{Det}_{A^1_{\phi_1}} \otimes \mathrm{Det}_{A^2_{\phi_2}} \otimes \mathrm{Det}_{\Sigma_2^0} & \xrightarrow{I\otimes\ell\otimes I} & \mathrm{Det}_{\Sigma_1^0} \otimes \mathrm{Det}_A \otimes \mathrm{Det}_{\Sigma_2^0}
\end{array} \qquad (5.17)$$

The transition function, is the composition of ℓ_P (determined by the top triangle) with the genus-zero trivialization maps

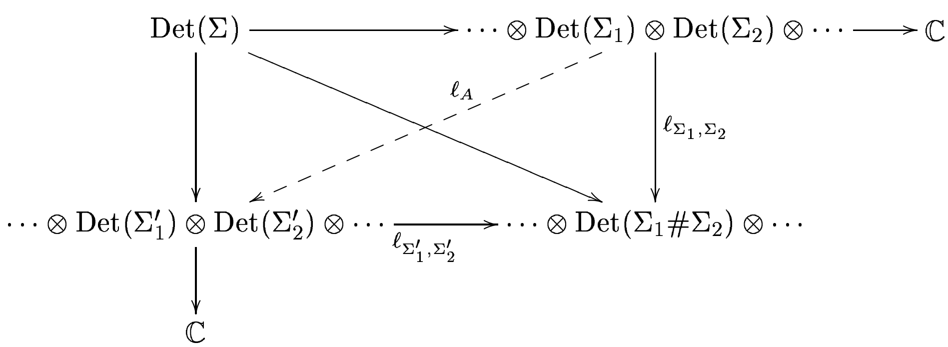
$$\mathrm{Det}_{\Sigma_1,\psi_1} \otimes \mathrm{Det}_{\Sigma_2,\psi_2} \to \mathbb{C} \quad \text{and} \quad \mathrm{Det}_{\Sigma_1,\phi_1} \otimes \mathrm{Det}_{\Sigma_2,\phi_2} \to \mathbb{C}.$$

By associativity of the sewing operation the diagram commutes and this implies that ℓ_P is equal to the composition of the four maps in the lower part of diagram. These maps are constructed from the sewing isomorphism of genus-zero surfaces and are thus holomorphic.

The next thing is to show the holomorphicity of the transition functions for pants decompositions that differ only by one elementary move.

Remark 5.6.2. We will change notation slightly at times by writing $\mathrm{Det}(\Sigma)$ instead of the usual Det_Σ. This is avoid an unnecessary number of subscripts.

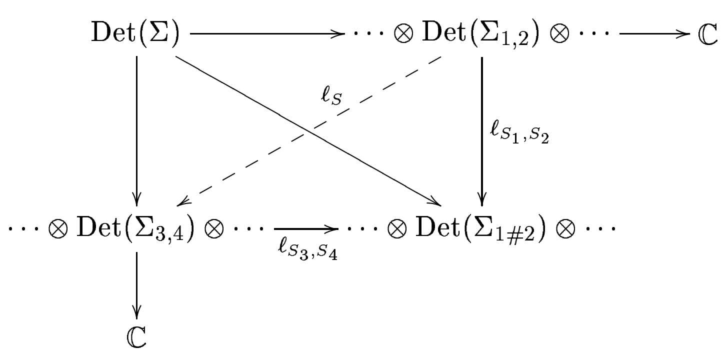
The A-move case: Let Σ be a 4-holed sphere. Let S_{12} and S_{34} be curves on Σ such that replacing S_{12} by S_{34} is an A-move. Let Σ_1 and Σ_2 be pants (3-holed spheres) obtained from Σ by cutting along S_{12} and let Σ_1' and Σ_2' be be pants obtained from Σ by cutting along S_{34}. The following diagram shows the trivialization obtained using the two different decompositions.



(5.18)

By associativity of the sewing operation each triangle commutes so the transition function from $\mathbb{C} \to \mathbb{C}$ can be expressed in terms of $\ell_A = \ell^{-1}_{\Sigma_1',\Sigma_2'} \circ \ell_{\Sigma_1,\Sigma_2}$ We know ℓ_{Σ_1,Σ_2} and $\ell_{\Sigma_1',\Sigma_2'}$ are holomorphic from the genus-zero theory and so ℓ_A is holomorphic.

The S-move case: Let Σ be a torus with one boundary component. Let S_{12} and S_{34} be cycles on Σ that intersect at one point. Replacing S_{12} with S_{34} is an S-move. Let $\Sigma_{1,2}$ and $\Sigma_{3,4}$ be the pants obtained from Σ by cutting along S_{12} and S_{34} respectively. The trivializations associated to these decompositions is shown in the following diagram.



(5.19)

As in the previous case each triangle commutes by associativity of the sewing operation. The transition function from $\mathbb{C} \to \mathbb{C}$ can be expressed in terms of $\ell_S = \ell^{-1}_{S_3,S_4} \circ \ell_{S_1,S_2}$.

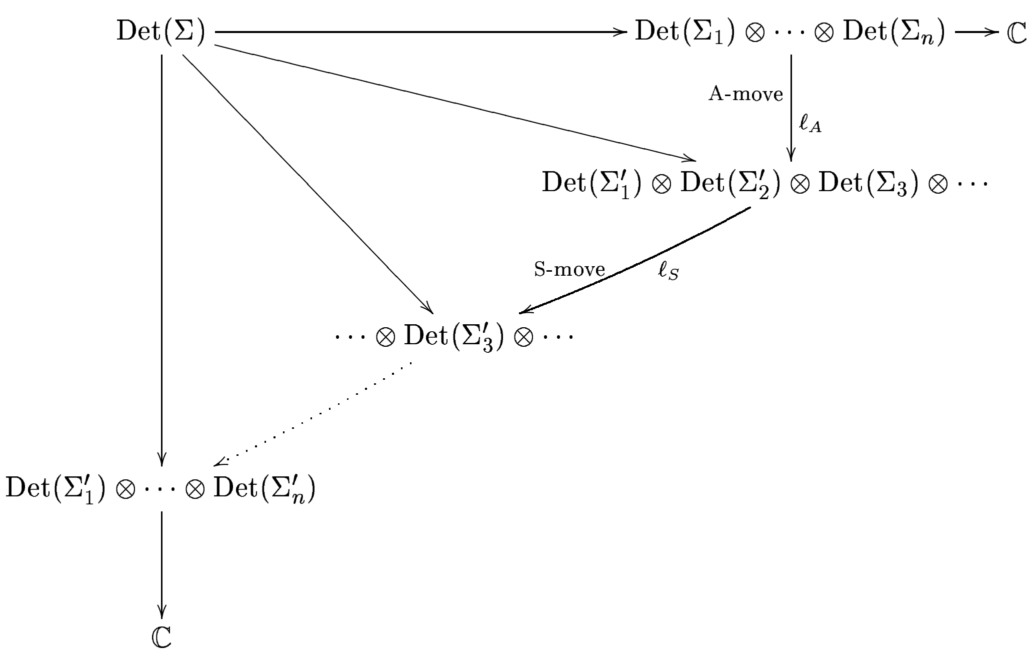
From Section 5.4, the self-sewing operations ℓ_{S_1,S_2} and ℓ_{S_3,S_4} are known to be holomorphic and thus ℓ_A is holomorphic.

Remark 5.6.3. The H-moves case: The following is not strictly needed as we have shown that S- and A-moves suffice to join any two pants decompositions. However, it shows that Corollary 5.5.7 is not required. If S and S' are homotopically equivalent then in the pants decomposition there are two possible cases.

1. Two surface Σ_1 and Σ_2 are effected by the change. In this case $\Sigma_1\#\Sigma_2$ is a sphere with four boundary components. Diagram 5.18 applies and the ℓ maps are still holomorphic as they are genus-zero sewing maps.

2. Two boundaries on a single surface, say $\Sigma_{1,2}$, are effected. The sewn surface $\Sigma_{1\#2}$ is a torus with one boundary component. Diagram 5.19 can be used and the holomorphicity follows from the same argument as in the S-move case.

The previous results will now be combined to obtain holomorphicity of a general transition function. As any two pants decompositions can be joined by a sequence of S- and A-moves, we can reduce any transition function to a sequence of transition functions for these moves. Consider the diagram



(5.20)

By associativity of the sewing operation all the triangles commute. So the transition function can be expressed as the composition of maps ℓ_A and ℓ_S, each of which we have shown to be holomorphic.

□

5.7 Holomorphicity of the sewing isomorphisms

There are two fundamental operations in the category of rigged surfaces. They are disjoint union and sewing. We need to show that both of these are holomorphic. In the case of disjoint union this is immediate. See Kriz [37] and Hu and Kriz [26] for details on the categorical structure of these operations.

The precise holomorphicity we want to prove is the holomorphicity of bundle maps. In the case of sewing two disjoint surfaces Σ_1 and Σ_2, we want to construct a holomorphic bundle map

$$\begin{array}{ccc} \mathrm{Det}_{\widetilde{T}_B(\Sigma_1 \sqcup \Sigma_2)} & \longrightarrow & \mathrm{Det}_{\widetilde{T}_B(\Sigma_1 \# \Sigma_2)} \\ \downarrow & & \downarrow \\ \widetilde{T}_B(\Sigma_1 \sqcup \Sigma_2) & \longrightarrow & \widetilde{T}_B(\Sigma_1 \# \Sigma_2). \end{array} \tag{5.21}$$

Note that we do not consider the case when $\Sigma_1 \# \Sigma_2$ has no punctures as we have not defined the determinant line for closed surfaces. To use the previous construction of the sewing isomorphism of determinant lines (see Section 5.2), and the holomophicity of the geometric sewing operation (see Section 4.5), we need to consider an additional bundle.

Remark 5.7.1. Given two vector bundles $E \to M$ and $F \to N$ a standard construction shows $E \otimes F$ is a vector bundle over the product space $M \times N$. This is called the *exterior tensor product.*

Using the exterior tensor product we consider the following bundle isomorphism.

$$\begin{array}{ccc} \mathrm{Det}_{\widetilde{T}_B(\Sigma_1)} \otimes \mathrm{Det}_{\widetilde{T}_B(\Sigma_2)} & \longrightarrow & \mathrm{Det}_{\widetilde{T}_B(\Sigma_1 \sqcup \Sigma_2)} \\ \downarrow & & \downarrow \\ \widetilde{T}_B(\Sigma_1) \times \widetilde{T}_B(\Sigma_2) & \longrightarrow & \widetilde{T}_B(\Sigma_1 \sqcup \Sigma_2) \end{array} \tag{5.22}$$

The map between the base spaces is the biholomorphism

$$\big([\Sigma_1, f, \Sigma_1', \phi_1], [\Sigma_2, g, \Sigma_2', \phi_2]\big) \mapsto [\Sigma_1 \sqcup \Sigma_2, f \sqcup g, \Sigma_1' \sqcup \Sigma_2', \phi_1, \phi_2].$$

The maps between the determinant lines is the canonical isomorphism

$$\mathrm{Det}_{\Sigma_1} \otimes \mathrm{Det}_{\Sigma_2} \simeq \mathrm{Det}_{\Sigma_1 \sqcup \Sigma_2}\,.$$

The holomorphicity of the bundle map follows directly from the definition of the determinant line.

Consider the bundle map

$$\begin{array}{ccc} \mathrm{Det}_{\widetilde{T}_B(\Sigma_1)} \otimes \mathrm{Det}_{\widetilde{T}_B(\Sigma_2)} & \longrightarrow & \mathrm{Det}_{\widetilde{T}_B(\Sigma_1 \# \Sigma_2)} \\ \downarrow & & \downarrow \\ \widetilde{T}_B(\Sigma_1) \times \widetilde{T}_B(\Sigma_2) & \longrightarrow & \widetilde{T}_B(\Sigma_1 \# \Sigma_2) \end{array} \tag{5.23}$$

The map between the base spaces is the geometric sewing map which was proved to be holomorphic in Section 4.5. The map between the fibers is the canonical sewing isomorphism ℓ_{Σ_1,Σ_2} (see Sections 5.1 and 5.2). If we can prove this bundle map is holomorphic, then combining Diagrams 5.23 and 5.22 gives holomorphic bundle maps

$$\begin{array}{ccccc} \mathrm{Det}_{\widetilde{T}_B(\Sigma_1 \sqcup \Sigma_2)} & \longrightarrow & \mathrm{Det}_{\widetilde{T}_B(\Sigma_1)} \otimes \mathrm{Det}_{\widetilde{T}_B(\Sigma_2)} & \longrightarrow & \mathrm{Det}_{\widetilde{T}_B(\Sigma_1 \# \Sigma_2)} \\ \downarrow & & \downarrow & & \downarrow \\ \widetilde{T}_B(\Sigma_1 \sqcup \Sigma_2) & \longrightarrow & \widetilde{T}_B(\Sigma_1) \times \widetilde{T}_B(\Sigma_2) & \longrightarrow & \widetilde{T}_B(\Sigma_1 \# \Sigma_2). \end{array}$$

This gives the holomorphic bundle map we sought to construct in Diagram 5.21. It remains to prove the holomorphicity of the bundle map in Diagram 5.23. This is where the real content of this section lies.

The holomorphic structure of the determinant line bundle was defined using pants decompositions. The decompositions are arbitrary and holomorphic transition functions relate different decompositions. In our situation we are concerned with the holomorphic structure of $\mathrm{Det}_{\widetilde{T}_B(\Sigma_1 \# \Sigma_2)}$. Since different pants decompositions give equivalent complex structures, we choose a decomposition in which the first step is cutting $\Sigma_1 \# \Sigma_2$ into Σ_1 and Σ_2. So the local coordinate chart is of the form

$$\mathrm{Det}_{\Sigma_1 \# \Sigma_2} \longrightarrow \mathrm{Det}_{\Sigma_1} \otimes \mathrm{Det}_{\Sigma_2} \longrightarrow, \ldots, \longrightarrow \mathbb{C}$$

and thus, by definition of the complex structure, the inverse of the sewing map

$$\mathrm{Det}_{\Sigma_1\#\Sigma_2} \longrightarrow \mathrm{Det}_{\Sigma_1} \otimes \mathrm{Det}_{\Sigma_2}$$

is holomorphic.

In the case of self sewing the situation is actually simpler as there is no tensor product to consider. The holomorphic bundle map is

$$\begin{array}{ccc} \mathrm{Det}_{\widetilde{T}_B(\Sigma_{1,2})} & \longrightarrow & \mathrm{Det}_{\widetilde{T}_B(\Sigma_{1\#2})} \\ \downarrow & & \downarrow \\ \widetilde{T}_B(\Sigma_{1,2}) & \longrightarrow & \widetilde{T}_B(\Sigma_{1\#2}) \end{array} \tag{5.24}$$

and the holomorphicity follows by an argument which is identical to the case of regular sewing considered above. We note again that the case when $\Sigma_{1\#2}$ has no punctures is not considered.

5.8 Determinant line bundle over the moduli space

In section 3.4 it was proved that the (pure) mapping class group acts fixed-point freely on the rigged Teichmüller space. The quotient space isomorphic to the moduli space of rigged surfaces. This isomorphism was used to define the complex manifold structure on the moduli space.

The determinant line bundle over the moduli space will be constructed in a similar fashion. To take the quotient, an action of the mapping class group on the determinant line bundle must be defined. Since the mapping class group is generated by Dehn twist it is enough to define the action of a Dehn twist. Dehn twists occur on annuli and so the sewing isomorphism and the genus-zero flat connection can be used to define the action.

Let $\rho \in \mathrm{PMod}(\Sigma)$ be a Dehn twist and recall that the action on $\widetilde{T}(\Sigma)$ is defined by

$$\rho \cdot [\Sigma, g, \Sigma_1, \phi] = [\Sigma, g \circ \rho, \Sigma_1, \phi].$$

The determinant lines are defined in terms of the canonical representatives so the action must be understood in terms of these surfaces. The canonical representative in

$[\Sigma, g, \Sigma_1, \phi]$ is $(\Sigma, f^{\mu_g}, \Sigma^{\mu_g}, \phi \circ g \circ (f^{\mu_g})^{-1})$, where $g \circ (f^{\mu_g})^{-1} : \Sigma_1 \to \Sigma^{\mu_g}$ is the biholomorphism that realizes the equivalence of triples. From now on we suppress the local coordinate. The canonical representative in $[\Sigma, g \circ \rho, \Sigma_1]$ is $(\Sigma, f^{\mu_{g\circ\rho}}, \Sigma^{\mu_{g\circ\rho}})$. To define the action of ρ on the determinant lines we need to produce an isomorphism

$$\mathrm{Det}\,\Sigma^{\mu_g} \longrightarrow \mathrm{Det}\,\Sigma^{\mu_{g\circ\rho}}$$

It is easier to work with the fixed underlying surface Σ and consider changes in complex structure. Recall that Σ_μ is the topological surface Σ with the μ-complex structure. The map

$$f^\mu : \Sigma_\mu \to \Sigma^\mu$$

is a biholomorphism and thus induces an isomorphism of determinant lines. Using this we have changed our problem to producing an isomorphism between the determinant lines of the surfaces Σ_{μ_g} and $\Sigma_{\mu_{g\circ\rho}}$. Let $A \subset \Sigma$ be the annulus such that the Dehn twist ρ occurs on the interior of A. Let $\rho(A)$ be the annulus after the application of the Dehn twist. The *canonical connection* for genus-zero surfaces constructed in Huang [30, Section D.4] gives an isomorphism

$$\rho_* : \mathrm{Det}\,A_{\mu_g} \longrightarrow \mathrm{Det}\,A_{\mu_{g\circ\rho}}.$$

On $\Sigma \setminus A$ the diffeomorphism ρ is the identity and thus the μ_g-complex structure and the $\mu_{g\circ\rho}$-complex structure are equivalent on $\Sigma \setminus A$. From this it follows that

$$\mathrm{Det}(\Sigma_{\mu_g} \setminus A) = \mathrm{Det}(\Sigma_{\mu_{g\circ\rho}} \setminus A).$$

Note that this is a real equality, not just an isomorphism.

Combining the above results leads to the sequence of isomorphisms

$$\begin{aligned} \mathrm{Det}\,\Sigma^{\mu_g} \longrightarrow \mathrm{Det}\,\Sigma_{\mu_g} \xrightarrow{\ell^{-1}} \mathrm{Det}(\Sigma_{\mu_g} \setminus A) \otimes \mathrm{Det}\,A_{\mu_g} &\xrightarrow{I\otimes\rho_*} \\ \xrightarrow{I\otimes\rho_*} \mathrm{Det}(\Sigma_{\mu_{g\circ\rho}} \setminus A) \otimes \mathrm{Det}\,A_{\mu_{g\circ\rho}} &\xrightarrow{\ell} \\ \xrightarrow{\ell} \mathrm{Det}\,\Sigma_{\mu_{g\circ\rho}} \longrightarrow \mathrm{Det}\,\Sigma^{\mu_{g\circ\rho}}. \end{aligned} \tag{5.25}$$

We define the action of the mapping class group on the determinant line bundle by this isomorphism which we also call ρ_*.

By general Teichmüller theory (see Appendix A) we know that f^{μ} depends holomorphically on μ. The sewing isomorphisms are holomorphic by Section 5.7. The other map, ρ_*, comes from the genus-zero flat connection which is also holomorphic by Huang [30]. We have thus proved the following result.

Proposition 5.8.1. *The mapping class group acts fixed-point freely as a group of biholomorphisms on the determinant line bundle over rigged Teichmüller space. For a Dehn twist $\rho \in \mathrm{PMod}(\Sigma)$, the action is defined by*

$$\rho \cdot \big([\Sigma, g, \Sigma_1, \phi], \lambda\big) = \big([\Sigma, g \circ \rho, \Sigma_1, \phi], \rho_*(\lambda)\big)$$

where $\lambda \in \mathrm{Det}\, \Sigma^{\mu_g}$, and ρ_* is defined by the isomorphism (5.25).

Let $\widetilde{\mathcal{M}}_B(\Sigma) = \widetilde{T}_B(\Sigma)/\mathrm{PMod}(\Sigma)$ be the moduli space corresponding to the Teichmüller space $\widetilde{T}_B(\Sigma)$. Using the action defined in Proposition 5.8.1 we define

$$\mathrm{Det}_{\widetilde{\mathcal{M}}_B(\Sigma)} = \mathrm{Det}_{\widetilde{T}_B(\Sigma)} / \mathrm{PMod}(\Sigma) \tag{5.26}$$

to be the *determinant line bundle over the moduli space* $\widetilde{\mathcal{M}}_B(\Sigma)$. Proposition 5.8.1 immediately gives us the result we have been aiming for.

Corollary 5.8.2. *The determinant line bundle, $\mathrm{Det}_{\widetilde{\mathcal{M}}_B(\Sigma)}$, over the moduli space $\widetilde{\mathcal{M}}_B(\Sigma)$ is a holomorphic line bundle.*

Chapter 6

Flat Connections and Modular Functors

The definition of a *modular functor* first appeared in Segal [48], and in Proposition 5.4 he states that: "*For any modular functor there is a canonical flat connection in the projective bundle of the bundle E_α on $\mathcal{C}_\alpha$ for every non-closed, labelled surface α. These connections are compatible with the sewing-together of surfaces.*"

The first aim of this chapter is to rigorize this statment with particular emphasis on the holomorphicity aspects of the definition of a (holomorphic) modular functor. Secondly we show that such a connection is determined by its restriction to the vector bundles over the moduli spaces of disks and annuli.

6.1 The tangent space to moduli space

As our constructions are local we can work directly with the moduli space of rigged surfaces instead of with Teichmüller space. Recall that Schiffer variation gives local coordinate charts for Teichmüller space. Local coordinates for the rigged Teichmüller can be obtained by Schiffer variation and variation of the local coordinates vanishing at the punctures. Because the action of the mapping class group is discrete and fixed-point free, neighborhoods of the moduli space can also be formed this way.

Recall that Schiffer variation with a complex parameter ϵ, of a surface Σ, produces a holomorphic family of surfaces Σ^ϵ. Let ϕ_t be an analytic family of local coordinates for Σ where t is a complex parameter. Let Σ_0 be the reference surface. Using the canonical representatives we have curves $[\Sigma_0, f^{\mu_\epsilon}, \Sigma^{\mu_\epsilon}, \phi]$ and $[\Sigma_0, f, \Sigma, \phi_t]$ in the moduli space $\widetilde{\mathcal{M}}_B(\Sigma)$. This proves the following lemma.

Lemma 6.1.1. *The tangent vectors to curves of this type form a basis for the tangent space to the moduli space $\widetilde{\mathcal{M}}_B(\Sigma)$. If Σ is of type (g, n) then $3g - 3 + n$ independent*

Schiffer variations need to be considered. The space of local coordinates in infinite-dimensional, so infinitely-many curves of the form ϕ_t must be used.

6.2 Central extensions of the semigroup of annuli

Following the notation in Segal [48], we define $\mathcal{A}$ to be the semigroup of the moduli space of annuli with analytically parametrized boundary components together with the sewing operation. Let $\mathcal{E}_A$ be a holomorphic line bundle over $\mathcal{A}$. Assume that there is a holomorphic bundle map

$$\ell : \mathcal{E}_A \otimes \mathcal{E}_A \longrightarrow \mathcal{E}_A$$

associated to sewing, such that $\mathcal{E}_A$ together with ℓ gives a central extension of $\mathcal{A}$. We now construct a connection on $\mathcal{E}_A$ that is compatible with sewing. In particular we construct such a connection for each central extension of $\mathrm{Diff}^+(S^1)$.

From the above central extension of $\mathcal{A}$, a central extension of $\mathrm{Diff}^+(S^1)$ can be obtained. See Segal [48] or Huang [30]. The classification of central extensions of $\mathrm{Diff}^+(S^1)$ states that the central extensions are determined by (c, h), where c is the *central charge* and h is the *weight*. It is also known (see Segal [48]) that the central extensions of $\mathcal{A}$ are determined, up to isomorphism, by their corresponding central extensions of $\mathrm{Diff}^+(S^1)$. If we can construct a central extension of $\mathrm{Diff}^+(S^1)$ of type (c, h) with a holomorphic flat connection then we can use this isomorphism to obtain a holomorphic flat connection on $\mathcal{E}_A$.

We construct a central extension of type (c, h) in two steps. The determinant line bundle $\mathrm{Det}_{\mathcal{A}}$ over $\mathcal{A}$ gives rise to a central extension of $\mathrm{Diff}^+(S^1)$ of type $(2, 0)$. Therefore $\mathrm{Det}_{\mathcal{A}}^{c/2}$ is of type $(c, 0)$, and we know that there is a holomorphic flat connection on this bundle.

We now construct a central extension of type $(0, h)$. First we work on the space of annuli of the form $\mathbb{A}_r^1$, where $r < 1$ and the boundaries are parametrized as follows. The outer circle is parametrized by $z \mapsto \alpha z$ for any $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. The inner circle is parametrized by $z \mapsto rz$. Since $r \in (0, 1)$ and $\alpha \in S^1$, this space of annuli can be identified with $(0, 1) \times S^1$.

For the given h, we can construct a line bundle over $(0,1)\times S^1$ with a flat connection such that locally $(r,\alpha)\mapsto((r,\alpha),\lambda\alpha^h)$, for any $\lambda\in\mathbb{C}$, is a flat section. We defer the construction of this bundle to the end of the proof. The sewing of this bundle must now be defined. It can be directly checked that sewing two annuli corresponding to (r_1,α_1) and (r_2,α_2) produces an annulus corresponding to $(r_1r_2,\alpha_1\alpha_2)$. Locally, any element in the fibers at (r_1,α_1) and (r_2,α_2) can be written as $((r_1,\alpha_1),\lambda_1\alpha_1^h)$ and $((r_2,\alpha_2),\lambda_2\alpha_2^h)$, respectively. We define the sewn element in the fiber at $(r_1r_2,\alpha_1\alpha_2)$ to be

$$((r_1r_2,\alpha_1\alpha_2),\lambda_1\lambda_2\alpha_1^h\alpha_2^h).$$

The semigroup of annuli, $\mathcal{A}$, is homotopically equivalent to the space of annuli corresponding to $(0,1)\times S^1$ considered above. Thus the line bundle and flat connection just constructed can be extended to a line bundle and flat connection on $\mathcal{A}$. Call this extended bundle M^h. The sewing operation also gives a sewing operation on M^h.

The line bundle M^h gives a central extension of $\mathcal{A}$, and the corresponding central extension of $\mathrm{Diff}^+(S^1)$ is of type $(0,h)$. The tensor product bundle $\mathrm{Det}_{\mathcal{A}}^{c/2}\otimes M^h$ thus gives a central extension of $\mathrm{Diff}^+(S^1)$ of type (c,h). Therefore the central extensions $\mathcal{E}_A$ and $\mathrm{Det}_{\mathcal{A}}^{c/2}\otimes M^h$ of $\mathcal{A}$ must be isomorphic. The flat holomorphic connection on $\mathrm{Det}_{\mathcal{A}}^{c/2}\otimes M^h$ induces a flat holomorphic connection on $\mathcal{E}_A$.

We now construct the bundle over $(0,1)\times S^1$ with the properties required above. Consider S^1 as the real line with points differing by $2\pi n$ identified for $n\in\mathbb{Z}$. Note that α^h is in general not a single valued function of $\alpha\in S^1$. Basically we want to construct a line bundle with flat connection such that locally α^h is a flat section. Take two open subsets $U_1=(-\pi/2,3\pi/2)$ and $U_2=(\pi/2,5\pi/2)$ of S^1. We will define the bundle to be trivial over U_1 and U_2 by taking single-valued branches of a^h on these two open subsets. Let the trivialization of the bundle over U_1 be given by the section $\alpha\mapsto\alpha^h$ where α^h is defined to be $e^{h\log\alpha}$ and the imaginary part of $\log\alpha$ is between $-\pi/2$ and π. Over the open subset U_2, we take the trivialization to be the one given by the section $\alpha\mapsto\alpha^h$, but here $\log h$ is chosen such that its imaginary part is between $\pi/2$ and $5\pi/2$. The intersection of U_1 and U_2 is the union of $(\pi/2,3\pi/2)$ and $(-\pi/2,\pi/2)$. The transition functions over these are given by 1 and $e^{2\pi ih}$, respectively. This gives us a line bundle together with a flat connection such that α^h is locally a flat connection. Denote the isomorphism between fibers induced from this connection by Z_A.

6.3 Modular functors

Let Φ be a finite set containing a distinguished element 1, and an involution $a \mapsto \bar{a}$. Let Σ a Riemann surface with ordered boundary components. A map from the set of boundary components of Σ to Φ is called a labelling of the boundary components. That is, each boundary component is assigned a label $a \in \Phi$. We write, $(\Sigma, a_1 \dots a_n)$ for the surface with ordered and labelled boundary components

Following Segal [48], we consider the category $\mathcal{C}_\Phi$ defined as follows. The objects are (possible disconnected) Riemann surfaces with analytically parametrized boundary components that are ordered and also labelled.

Remark 6.3.1. In Segal [48], the boundary components are not ordered. This additional data is natural to add in the process of constructing modular functors.

Morphisms in $\mathcal{C}_\Phi$ are the sewing operations as well as biholomorphisms of the surfaces that preserve the extra data. The inclusion of biholomorphisms is to account for the case of sewing empty subsets of boundary components. Let $\mathcal{V}$ be the category of finite dimensional vector spaces whose morphisms are linear maps (not necessarily isomorphisms).

Note that we can define the rigged Teichmüller space for surfaces with the extra data of the labelled boundaries in a straightforward way. See Chapter 3 and Appendix A for background and notation. With a slight abuse of notation we will use $\widetilde{T}_B(\Sigma)$ for this Teichmüller space. The corresponding moduli space will be denoted $\widetilde{\mathcal{M}}_B(\Sigma)$ as before. Recall that Σ^μ, or more precisely $[\Sigma, f^\mu, \Sigma^\mu]$, is the canonical representative of the Teichmüller equivelnce class. In our case Σ^μ must also be labelled.

We now give a definition of a holomorphic modular functor.

Definition 6.3.1. A *holomorphic modular functor* is a functor, E, from $\mathcal{C}_\Phi$ to $\mathcal{V}$ satisfying the following axioms.

1. The vector spaces $E(\Sigma^{\mu})$ form holomorphic vector bundles over the rigged Teichmüller spaces $\widetilde{T}_B(\Sigma)$.

2. There is a holomorphic action of pure mapping class group on these vector bundles. (The corresponding quotient thus produces holomorphic vector bundles over the moduli spaces $\widetilde{\mathcal{M}}_B(\Sigma)$, of Riemann surfaces with ordered, labelled and parametrized boundaries, which we have shown to be a complex manifold in Theorem 3.4.4.)

3. $E(\Sigma \sqcup \Sigma') \simeq E(\Sigma) \otimes E(\Sigma')$ (where "$\simeq$" denotes a natural isomorphism between functors).

4. If Σ is cut along a curve to produce a new (possibly disconnected) surface (Σ', aa), with the two new boundary components both labelled by a, then there is a natural isomorphism
$$\ell : \bigoplus_{a\in\Phi} E(\Sigma', aa) \longrightarrow E(\Sigma)$$
which is holomorphic as an operation between the corresponding vector bundles, and satifies the obvious associativity axiom as in the determinant line bundle case (see Chapter 5).

5. If D is a disk, then
$$\dim E(D,a) = \begin{cases} 1 & \text{if } a = 1 \\ 0 & \text{otherwise.} \end{cases}$$

6. Let (A, ab) be an annulus with boundary components labelled by a and b. If the boundary components are oppositely oriented then
$$\dim E(A,ab) = \begin{cases} 1 & a = b \\ 0 & \text{otherwise.} \end{cases}$$
If the boundary components have the same orientation then
$$\dim E(A,ab)) = \begin{cases} 1 & a = \bar{b} \\ 0 & \text{otherwise.} \end{cases}$$
In particular the $E(A, aa)$ form a central extension of $\mathcal{A}$.

7. There are holomorphic projectively flat connections on these vector bundles which are compatible with the sewing operation. By compatibility with the sewing we mean that diagrams of the form

$$\begin{array}{ccc}
E(\Sigma) & \xrightarrow{\ \ell^{-1}\ } & \bigoplus_{a\in\Phi} E(\Sigma_1,a)\otimes E(A,aa)\otimes E(\Sigma_2,a) \\
 & & \downarrow{\scriptstyle I\otimes Z_A\otimes I} \\
 & & \bigoplus_{a\in\Phi} E(\Sigma_1,a)\otimes E(A^t,aa)\otimes E(\Sigma_2,a) \\
 & \searrow{\scriptstyle L} & \downarrow{\scriptstyle \ell} \\
 & & E(\Sigma^t)
\end{array}$$

commute, where $\Sigma^t = \Sigma_1 \# A^t \# \Sigma_2$, A^t is a family of annuli, Z_A is the isomorphism constructed in Section 6.2, and L is the isomorphism induced by the given connections.

This may not be the minimal set of axioms but we are not concerned with these issues. Whichever definition is chosen the above properties are true and are the properties we need.

6.4 Connections

Let $\mathcal{M}_K$ (repectively, $\mathcal{M}_A$) be the moduli space of disks (respectively, annuli) with analytically parametrized and labelled boundaries. Let $\mathcal{E}_K$ and $\mathcal{E}_A$ be the holomorphic vector bundles over $\mathcal{M}_K$ and $\mathcal{M}_A$ given by the modular functor. Let L_K (respectively, L_A) be the isormorphisms bewtween fibers of $\mathcal{M}_K$ (respectively, $\mathcal{M}_A$) determined by the connections. Let $\mathcal{E}_\Sigma$ be the vector bundle over the moduli space $\widetilde{\mathcal{M}}_B(\Sigma)$ given by the modular functor. The isomorphism between surfaces induced by the connection will be denoted L.

Since any disk can be decomposed as a given disk sewn with an annulus, any connection on $\mathcal{E}_A$ induces a connection on $\mathcal{E}_K$. In particular the connection Z_A induces a connection, Z_K, on $\mathcal{E}_K$.

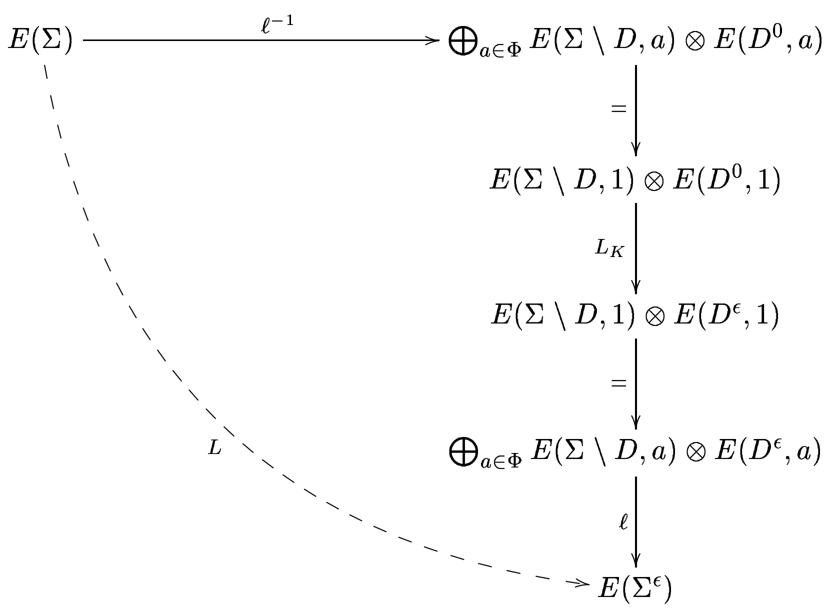
Theorem 6.4.1. *The connections L_K and L_A, on $\mathcal{M}_K$ and $\mathcal{M}_A$, agree with the connections Z_K and Z_A constructed above.*

Proof. The statement follows from the compatibility of the connection with the sewing operation. □

Theorem 6.4.2. *Any holomorphic projectively flat connection on the vector bundle $\mathcal{E}_\Sigma$ over the moduli space is determined by the restriction of the connection to the vector bundles over $\mathcal{E}_K$ and $\mathcal{E}_A$.*

Proof. Following Lemma 6.1.1 we consider a curve of the form $[\Sigma, f^{\mu_\epsilon}, \Sigma^{\mu_\epsilon}, \phi]$. Instead of working directly with this family we work with the Schiffer family Σ^ϵ. Let D be the disk on Σ where the Schiffer variation is performed and let D^ϵ be the deformed unit disk in the complex plane as in Section 2.6.

The following diagram is commutative because of the compatibility of the connection with sewing and Theorem 6.4.1:



This diagram produces an isomorphism relating surfaces along a path created by Schiffer variation. Thus we see that L is determined by L_K.

The other type of path can be dealt with in a similar way. Let S be a boundary component of Σ and ϕ_t a family of parametrizations. We cut Σ along a curve C that is homotopic to S, and choose a parametrization of the resultant boundary.

Let A be the annular region on Σ bounded by C and S. We use the notation (A, ab, ϕ_t) to specify the labelling by ab, and the parametrization of S by ϕ_t. As defined above there is an isomorphism $L_A : E(A, \phi_0) \to E(A, \phi_t)$. determined by the connection. Combining this with the sewing isomorphism we again get a sequence of isomorphisms that relate the two surfaces.

Let (Σ, b, ϕ) denote a surface with one boundary labelled by b and parametrized by ϕ. The compatibility of the connections with the sewing operations means we have a commutative diagram

$$
\begin{array}{ccc}
E(\Sigma, b, \phi_0) & \xrightarrow{\ell^{-1}} & \bigoplus_{a\in\Phi} E(\Sigma \setminus A, a) \otimes E(A, ab, \phi_0) \\
 & & \downarrow = \\
 & & E(\Sigma \setminus A, b) \otimes E(A, bb, \phi_0) \\
 & & \downarrow L_A \\
 & & E(\Sigma \setminus A, b) \otimes E(A, bb, \phi_t) \\
 & & \downarrow = \\
 & L & \bigoplus_{a\in\Phi} E(\Sigma \setminus A, b) \otimes E(A, ab, \phi_t) \\
 & & \downarrow \ell \\
 & \dashrightarrow & E(\Sigma, b, \phi_t).
\end{array}
$$

We see that L is determined by L_A.

□

By Theorems 6.4.1 and 6.4.2 and Section 6.2 we have the following:

Corollary 6.4.3. *The connections L_K and L_A are determined by a central charge c and a set of weights h_a, indexed by a in the labelling set I.*

Appendix A

Quasiconformal Mappings

A.1 Introduction

Quasi-conformal maps between Riemann surfaces is a class of homeomorphisms that arise naturally, and in fact necessarily, in the study of Teichmüller spaces. There are many equivalent definitions of "quasi-conformal" but essentially there are only two. The first was a geometric generalization of conformal mappings by Grötzsch in the 1930's. The modern analytic definition was brought to prominence by Ahlfors in the early 1950's. He was also the first to use the term "quasiconformal".

Initially quasiconformal maps were used as a tool to prove and generalize facts in complex analysis. Then Teichmüller made the deep discovery of the connection between extremal problems in quasiconformal mappings and Riemann surfaces. In the 1950's Ahlfors clarified and reproved the results of Teihmüller. This not only popularized Teichmüller's ideas, but also brought quasiconformal mappings into mainstream mathematics. One of the key ideas was to remove the differentiability condition from the definition of "quasiconformal".

The geometric idea is the following. Under a conformal map, small circles are mapped to circles. Roughly speaking, quasiconformal maps are those homeomorphisms which deform small circles into ellipses with the condition that the ratio of the minor axis to major axis remain globally bounded.

The analytic approach comes from generalizing the concept of conformality by looking at the Cauchy-Riemann equations. In terms of variables z and $\bar{z}$, the Cauchy-Riemann equations are $\frac{\partial f}{\partial \bar{z}} = 0$. A function, f, is analytic if and only if $\frac{\partial f}{\partial \bar{z}} = 0$. The appropriate generalization of this is the partial differential equation $\frac{\partial f}{\partial \bar{z}} = \mu(z)\frac{\partial f}{\partial z}$, called the Beltrami equation. Note that if $\mu = 0$ then this reduces to the Cauchy-Riemann equations. If f is a solution to this equation for $\mu \in L^\infty(\mathbb{C})$ with $||\mu|| < 1$ then f is quasiconformal.

The derivatives above are locally L^p-integrable distributional derivatives. A full investigation of quasiconformal maps requires a serious use of real analysis. Deducing the regularity properties from the geometric definition, and hence the proof of the equivalence of the definitions, took several years.

Key References:

- [39] Lehto and Virtanen, "Quasiconformal mappings of the plane". This is the main resource for detailed proofs.
- [44] Nag, "The Complex analytic Theory of Teichmüller Spaces". Extensive treatment containing important definitions and theorems. Not all details of the proofs are given but references are always included. Intuition and outline of proofs are mostly included.
- [1] Ahlfors, "Lectures on Quasiconformal Mapping". A classic reference containing rigorous proofs.

Other References:

- [38] Krushkal, "Quasiconformal Mappings and Riemann Surfaces". Translated from Russian by I.Kra.
- [11] "Quasiconformal Mappings and Analysis, A collection of papers honoring F.W.Gehring". Some interesting history and papers by major contributors to the field.

A.2 Basic definitions and results

In this section we collect the results on quasiconformal mappings that are directly needed in understanding the complex manifold structure of Teichmüller space. We follow Nag [44, Chapter 1]. This treatment is standard and essentially follows the classic book of Lehto and Virtanen [39], but adds the material need for Riemann Surfaces.

Let w be an orientation preserving homoeomorphism between planar domains D_1 and D_2. Let

$$\begin{aligned} L(z,r) &= \max_{\xi}\{|w(\xi)-w(z)| : |\xi - z| = r\} \\ l(z,r) &= \min_{\xi}\{|w(\xi)-w(z)| : |\xi - z| = r\} \end{aligned}$$

for r small and positive.

Definition A.2.1. The *circular dilation* at z is the function

$$H(z) = \limsup_{r\to 0^+} \frac{L(z,r)}{l(z,r)}$$

Definition A.2.2. The homeomorphism w is called *quasiconformal* if the circular dilation $H(z)$ is globally bounded on D_1.

If $H(z) \leq K$ almost-everywhere on D_1 then w is called K-quasiconformal. Note that such a K always exists and $K \geq 1$.

Definition A.2.3. If w is differentiable at z and $\mathrm{Jac}(w)$ is positive then

$$\mu_w(z) = \frac{\frac{\partial w(z)}{\partial \bar{z}}}{\frac{\partial w(z)}{\partial z}}$$

is called the *complex dilation* of w at z. Such a point, z, is called a *regular point.*

Note that $\mu = \mu_w(z)$ is a well-defined complex number and $|\mu| < 1$. Some easy calculations show that if w is C^1 then $H(z) = \frac{1+|\mu|}{1-|\mu|}$.

Proposition A.2.1. *If w is a quasiconformal map on D_1 then almost every point of D_1 is a regular point and thus the complex dilation $\mu_w(z)$ is defined almost-everywhere on D_1.*

For more details on the geometric definition see Lehto and Virtanen [39, Chapters 1 and 2]. In fact the definitions there are given in terms of the module of quadrilaterals. This seems to be the more basic geometric concept. The circular dilation is only introduced in Chapter 2.

We now give the analytic definition of a quasiconformal map. Again there are various definitions. Of course there is flexibility in what is a definition and what is a theorem.

Definition A.2.4. An orientation-preserving homoeomorphism w on a domain $D \subset \mathbb{C}$ is called *quasiconformal* if w has locally L^p-integrable distributional derivatives on D for some $p \geq 1$ and satisfies

$$|\frac{\partial w(z)}{\partial \bar{z}}| \leq k|\frac{\partial w(z)}{\partial z}|$$

almost everywhere on D for some $k \in [0, 1)$. The map w is called K-quasiconformal for any $K \geq (1+k)/(1-k)$.

A proof of the equivalence of the definitions of "quasiconformal" can be found in Lehto and Virtanen [39, page 168]. If w is a C^1-diffeomorphism then the equivalence is easy to see.

Proposition A.2.2. *A homeomorphism w on D is quasiconformal if and only if w has locally L^1 generalized derivatives on D satisfying*

$$\frac{\partial w(z)}{\partial \bar{z}} = \mu(z)\frac{\partial w(z)}{\partial z} \tag{A.1}$$

almost everywhere on D for some measurable function μ on D with $\|\mu\|_\infty < 1$. If $\|\mu\|_\infty = k < 1$, then w is K-quasiconformal for every $K \geq (1+k)/(1-k)$.

This follows easily from the analytic definition once it is proved that $\frac{\partial w}{\partial z} \neq 0$ almost everywhere. An outline is in Nag [44] and the details are in Lehto and Virtanen [39]. For w quasiconformal the smallest K is

$$K(w) = \frac{1 + \|\mu\|_\infty}{1 - \|\mu\|_\infty}$$

and is called the *dilation* of w.

The partial differential equation A.1, is called the *Beltrami equation* with coefficient μ. Any w satisfying this equation will be called a *μ-conformal homeomorphism on D.*

It is easy to see that for $\mu = 0$, that the Beltrami equation reduces to the Cauchy-Riemann equations and hence (by Weyl's lemma) w is analytic. So we see that a 1-quasiconformal homeomorphism is in fact conformal. The dilation in this case is 1.

The first natural questions to arise are those of existence and uniqueness. Both are answered in the affirmative (See Theorem A.2.5) although uniqueness is only up to normalization of three points corresponding to the fact that μ can be composed with any conformal map.

Theorem A.2.3. *The inverse of a K-quasiconformal homeomorphism is also a K-quasiconformal homeomorphism. The composition of a K_1-quasiconformal homeomorphism with a K_2-quasiconformal homeomorphism is a K_1K_2-quasiconformal homeomorphism*

These assertions follow easily from the geometric definition but are hard to prove using the analytic definition.

The formula for the complex dilation of a composition of two quasiconformal homeomorphism is extremely important and will be used repeatedly. For smooth maps it follows just by usual chain-rule computations. The general case is more difficult. Let $f : D' \to D$ and $w : D \to w(D)$ be quasiconformal homeomorphisms. The complex dilation of $(w \circ f)$ is given by

$$\mu_{w\circ f} = \frac{\mu_f + (\mu_w \circ f) r_f}{1 + (\mu_w \circ f) r_f \bar{\mu}_f}$$

almost everywhere on D', where $r_f = \frac{\partial \bar{f}}{\partial \bar{z}} / \frac{\partial f}{\partial z}$

If w is conformal and f is quasiconformal then from Equation A.2 we easily see that

$$\mu_{w\circ f} = \mu_f.$$

So composing on the left by a conformal map does not change the complex dilation. Note that this is not true for right composition. A direct computation also shows that if $w \circ f = h$, for quasiconformal w and f, then

$$(\mu_{h\circ f^{-1}}) \circ f = (1/r_f)(\mu_h - \mu_f)/(1 - \mu_h)\bar{\mu}_f$$

almost everywhere.

For a domain $D \subset \mathbb{C}$, let $L^\infty(D)_1$ be the unit ball in $L^\infty(D)$. This unit ball is of interest because of the condition $\|\mu\|_\infty < 1$.

The following theorem states the “uniqueness” of the Beltrami equation.

Theorem A.2.4. *If f and h are two quasiconformal homeomorphisms on D satisfying the same Beltrami equation, for some $\mu \in L^\infty(D)_1$, then $h \circ f^{-1}$ and $f \circ h^{-1}$ are biholomorphisms. Conversely, for any conformal mapping σ on $f(D)$, $\sigma \circ f$ satisfies the same Beltrami equation as f.*

Proof. This follows directly from the formulae above. $\square$

We now turn to the existence question. Any $\mu \in L^\infty(D)_1$ can extend to $L^\infty(\mathbb{C})_1$ by setting it to zero on $\mathbb{C}\backslash D$. So we only need to solve the existence problem for arbitrary $\mu \in L^\infty(\mathbb{C})_1$.

In what follows we are thinking of $\mathbb{C}$ as $\hat{\mathbb{C}}\backslash\{\infty\}$. It can be shown (Lehto and Virtanen [39]) that, as for conformal maps, isolated boundary points are removable singularities. From this it follows that any quasiconformal homeomorphism with domain $\mathbb{C}$ must map onto $\mathbb{C}$. So any quasiconformal map can be extended to $\hat{\mathbb{C}}$ sending ∞ to ∞. Let $f : \mathbb{C} \to \mathbb{C}$ be a quasiconformal homeomorphism. By composing with conformal automorphisms of $\mathbb{C}$ we can normalize f by fixing the action on two points. Say $f(0) = 0$ and $f(1) = 1$. Then we can extend f to $\hat{\mathbb{C}}$ by setting $f(\infty) = \infty$.

With this motivation we state the crucial existence theorem for the Beltrami equation.

Theorem A.2.5. *Given any $\mu \in L^\infty(\mathbb{C})_1$, there exists a unique μ-conformal homeomorphism $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ fixing $0, 1$ and ∞. This quasiconformal f will be denoted w^μ.*

The proof of this result is hard. Details can be found in Ahlfors [1] and Lehto and Virtanen [39]. The outline in Nag [44] is instructive in that one sees fairly directly the analytic dependence of w^μ on μ. This analyticity is vital in understanding the complex analytic structure on Teichmüller space and is stated formally in the following theorem.

Theorem A.2.6. *For every fixed $z \in \mathbb{C}$ the map $\mu \mapsto w^\mu$ from $L^\infty(\mathbb{C})_1$ to $\mathbb{C}$ is holomorphic.*

Recall that U and L are the upper and lower half-planes respectively. Another map related to w^μ that only depends real analytically on μ will now be defined. For $\mu \in L^\infty(U)_1$, define $\mu^* \in L^\infty(\mathbb{C})_1$ by

$$\mu^*(z) = \begin{cases} \mu(z) & \text{for } z \text{ in } U, \\ \overline{\mu(\bar{z})} & \text{for } z \text{ in } L. \end{cases}$$

By the existence theorem we have a normalized quasiconformal map w^{μ^*}. This map in fact preserves U and L. Let w_μ be the restriction of w^{μ^*} to U. Note that w_μ is the unique μ-conformal self-homeomorphism of U whose extension to $\hat{\mathbb{R}}$ fixes $0, 1$, and ∞. It is worth emphasizing that w^μ differs from w_μ in that μ is extended to $\mathbb{C}$ by setting it equal to 0 on L.

A.3 Quasiconformal maps, Riemann surfaces and Fuchsian groups

Quasiconformal maps can be used to produce new Riemann surfaces. Through the representation of Riemann surfaces by Fuchsian groups, Tiechmüller space can in fact be parametrized by certain equivalence classes of Beltrami differentials. This section provides the basic definitions and results of this theory. The material has been taken from Nag [44].

A.3.1 Quasiconformal mappings of Riemann surfaces

An orientation preserving homeomorphism between two Riemann surfaces is called quasiconformal if is locally quasiconformal in the planar sense.

Let κ be the holomorphic cotangent bundle over the Riemann surface Σ. A tensor of type (p, q) is an assignment of a function $\mu(z)$ to each coordinate chart on Σ such that $\mu(z)dz^p d\bar{z}^q$ is invariant under local holomorphic coordinate changes. Equivalently, a (p, q) tensor on Σ is a section of the complex line bundle $\xi = \kappa^p \otimes \bar{\kappa}^q$ over Σ.

If f is a quasiconformal homeomorphism between two Riemann surfaces then the *complex dilation* $\mu_f(z) = f_{\bar{z}}/f_z$ is a $(-1, 1)$ tensor on Σ. This follows from the formula for the complex dilation of a composition. In fact μ_f is in the unit open ball of the Banach space of sections $L^\infty(\kappa^{-1} \otimes \bar{\kappa})$. We denote this unit ball by $L^\infty(\kappa^{-1} \otimes \bar{\kappa})_1$ or $L^\infty_{(-1,1)}(\Sigma)_1$

Given any $\mu \in L^\infty(\kappa^{-1} \otimes \bar{\kappa})_1$ we can produce a complex analytic atlas on Σ whose local charts are quasiconformal with respect to the original complex structure. We call this the "μ complex structure" on Σ. The surface Σ with this new complex structure will be written Σ_μ. The identity map $1 : \Sigma \to \Sigma_\mu$ has complex dilation μ.

A.3.2 Representation by Fuchsian groups

For background on Fuchsian groups see for example Nag [44] or Farkas and Kra [12]. For our purposes the important result is the following.

Proposition A.3.1. *A torsion-free, finitely generated, first-kind Fuchsian group G will produce a Riemann surface $\Sigma = U/G$ of finite conformal type (g, n) for all possible (g, n) except $(0,0), (0,1), (0,2)$ and $(1,0)$.*

The condition $2g - 2 + n > 0$ excludes precisely the exceptional cases. In our main work this condition is always imposed.

Let G be the Fuchsian group uniformizing Σ. That is, $\Sigma = U/G$. We will produce a group to uniformize Σ_μ by conjugating with a quasiconformal homeomorphism.

Lemma A.3.2. *Let G be any Kleinian group. If w is a quasiconformal homeomorphism of $\hat{\mathbb{C}}$ with complex dilation μ, then wGw^{-1} is a group of möbius transformations if and only if $(\mu \circ g)\overline{g'}/g' = \mu$ almost everywhere on $\hat{\mathbb{C}}$ for each $g \in G$*

If w satisfies the conditions in the lemma then we say that w is *compatible* with G. Let Ω be the region of discontinuity for G and Ω_1 any invariant subregion. Let $L^\infty(\Omega_1, G) = \{\mu \in L^\infty(\Omega_1) \,|\, (\mu \circ g)\overline{g'}/g' = \mu$ a.e. in Ω_1 for all g in $G\}$. For $\mu \in L^\infty(\Omega_1, G)$, let w^μ be the normalized solution of the Beltrami equation with coefficient μ on Ω_1 and zero elsewhere. Note that w^μ is compatible with G.

Definition A.3.1. We call the deformed Kleinian group

$$G^\mu = w^\mu G (w^\mu)^{-1}$$

a quasiconformal deformation of G supported on Ω_1. Similarly we define

$$G_\mu = w_\mu G (w_\mu)^{-1}.$$

We now specialize to Fuchsian groups $G \subset \text{Möb}(\mathbb{R})$. With $\Omega_1 = U$ we have the space $L^\infty(U, G) = \{\mu \in L^\infty(U) \,|\, (\mu \circ g)\overline{g'}/g' = \mu$ a.e. in U for all g in $G\}$.

Proposition A.3.3. *The element $g^\mu = w^\mu g (w^\mu)^{-1} \in \text{Möb}(\mathbb{C})$ depends complex analytically on $\mu \in L^\infty(U, G)_1$ for any fixed g.*

Note that the dependence of $g_\mu = w_\mu g(w_\mu)^{-1}$ on μ is only real analytic.

Proposition A.3.4. *If G is a torsion-free Fuchsian group and $\Sigma = U/G$, then the Banach spaces $L^\infty(U,G)$ and $L^\infty_{(-1,1)}(\Sigma)$ are canonically isometrically isomorphic.*

This is in fact true for more general Fuchsian groups (see Nag [44, page 50]).

The following theorem is fundamental and will be used in several ways.

Theorem A.3.5. *Let Σ be any Riemann surface and let $\mu \in L^\infty_{(-1,1)}(\Sigma)_1$. Let G be the Fuchsian group (possibly with torsion) such that $\Sigma = U/G$. Then the Riemann surface Σ_μ is canonically biholomorphic to each of U/G_μ and $w^\mu(U)/G^\mu$. The biholomorphism extends to map any punctures that may be present to punctures.*

Outline of Proof. The quasiconformal map $w^\mu : U \to w^\mu(U)$ induces the homeomorphism $(w^\mu)_* = f^\mu : U/G \to w^\mu/G^\mu$. This map is a biholomorphism with respect to the μ-complex structure on Σ. It is also important to note that the projections $\pi : U \to \Sigma$ and $\pi^\mu : U \to w^\mu/G^\mu$ are holomorphic (branched covering spaces). □

A.3.3 Teichmüller spaces and Fuchsian groups

Let $G \subset \text{Möb}(\mathbb{R})$ be an Fuchsian group.

Definition A.3.2. The Teichmüller space of G is

$$T(G) = L^\infty(U,G)_1/\sim$$

where $\mu \sim \nu$ if and only if $w_\mu|_\mathbb{R} = w_\nu|_\mathbb{R}$

The Riemann moduli space of G is

$$R(G) = L^\infty(U,G)_1/\sim_R$$

where $\mu \sim_R \nu$ if and only if G_μ and G_ν are conjugate in $\text{Möb}(\mathbb{R})$.

Definition A.3.3. Let

$$Q(G) = \{w \text{ q.c. homeo of } U \,|\, wGw^{-1} \subset \text{Möb}(\mathbb{R})\},$$

$$Q_n(G) = \{w \in Q(G) \,|\, w \text{ fixes each of } 0,1,\infty\},$$

$$Q_0(G) = \{w \in Q_n) \,|\, w|_\mathbb{R} = 1|_\mathbb{R}\}.$$

It is worth noting that $L^\infty(U,G)_1$ is in one-to-one correspondence with $Q_n(G)$.

The next result connects the definitions of Teichmüller space and Teichmüller group. We have not discussed the metric on the Teichmüller group as it is not explicitly need for our purposes.

Theorem A.3.6. *For any non-elementary torsion-free Fuchsian group G, with $U/G = \Sigma$, there is a canonical isometry $\rho : T(G) \to T(\Sigma)$ given by $\rho([\mu]) = [\Sigma, f_\mu, U/G_\mu]$.*

Lemma A.3.7. *The image of w^μ is independent of choice of representative of $[\mu] \in T(G)$. That is, if $[\mu] = [\nu]$ then $w^\mu(U) = w^\nu(U)$*

See Nag [44, page 189] for a proof. It is important to realize that it is not true that the maps are independent of equivalence class representative. In fact it is generally the case that $w^\mu(z) \neq w^\nu(z)$ for $z \in U$.

The results of the next lemma and corollary are not explicitly stated in Nag.

Lemma A.3.8. *If $[\mu] = [\nu]$ in then $[\Sigma, f^\mu, \Sigma^\mu] = [\Sigma, f^\nu, \Sigma^\nu]$ in $T(\Sigma)$.*

Proof. We already know from Theorem A.3.6 that $[\Sigma, f_\mu, U/G_\mu] = [\Sigma, f_\nu, U/G_\nu]$. Theorem A.3.5 implies that the map

$$f^\mu \circ f_\nu^{-1} : U/G_\mu \longrightarrow \Sigma^\mu$$

is a biholomorphism. Therefore $[\Sigma, f_\mu, U/G_\mu] = [\Sigma, f^\mu, \Sigma^\mu]$ via the biholomorphism $\sigma = f^\mu \circ f_\nu^{-1}$ and so

$$[\Sigma, f^\mu, \Sigma^\mu] = [\Sigma, f_\mu, U/G_\mu] = [\Sigma, f_\nu, U/G_\nu] = [\Sigma, f^\nu, \Sigma^\nu].$$

□

Using this lemma we can formulate a result analogous to that of Theorem A.3.6.

Corollary A.3.9. *With G as in Theorem A.3.6 there is a canonical isometry $T(G) \to T(\Sigma)$ given by $[\mu] \mapsto [\Sigma, f^\mu, \Sigma^\mu]$.*

The proof follows directly from Lemma A.3.8 and the proof of Theorem A.3.6 given in Nag [44, page 120].

Recall that the modular (or mapping class) group $\text{Mod}(\Sigma) = Q(\Sigma)/Q_0(\Sigma)$ acts as a group of isometries on $T(\Sigma)$. An analogous procedure can be carried out for the Tecimüller groups $T(G)$.

Definition A.3.4. For any $w \in Q(G)$ an *allowable bijection* is the map $w^* : T(wGw^{-1}) \to T(G)$ defined by sending $[\mu] \in L^\infty(U, wGw^{-1})$ to $[w_\mu \circ w]$.

In fact w^* is an isometry and if $wGw^{-1} = G$ then w^* is a self-isometry. These w form the normalizer subgroup of G in $Q(G)$. Let

$$N_{\text{q.c}}(G) = \{w \in Q(G) \,|\, wGw^{-1} = G\}$$

The action of w^* depends only on its boundary values so we identify elements of $N_{\text{q.c.}}$ that differ only by composition with an element of $Q_0(G)$.

Definition A.3.5. The *extended modular group of* G is

$$\text{mod}(G) = N_{\text{q.c.}}(G)/(N_{\text{q.c.}}(G) \cap Q_0(G))$$

Any $g \in G$ can be consider an element of $N_{\text{q.c.}}(G)$. Then $g^* : T(G) \to T(G)$ is the identity.

Definition A.3.6. The *modular groups of* G is

$$\text{Mod}(G) = \text{mod}(G)/G.$$

If G is a torsion-free Fuchsian group then $\text{Mod}(G)$ is Riemann surface U/G. Moreover, $R(G) = T(G)/\text{Mod}(G)$.

A.3.4 Teichmüller's theorem

Here we follow the discussion of Nag [44, Section 2.6]. Teichmüller's theorem is fundamental to the construction of the complex structure on Teichmüller space. In Chapter 3 we have a different use the result in quite a different way.

Recall that subtle point that although the spaces $w^\mu(U)$ and Σ^μ do not depend on the choice of representative, the maps w^μ and $f^\mu = (w^\mu)_*$ do. So $(\Sigma, f^\mu, \Sigma^\mu)$ is not a canonical representative of its Teichmüller equivalence class.

Given a quasiconformal homeomorphism $f : \Sigma \to \Sigma_1$, *Teichmüller's extremal problem* is to find a quasiconformal homeomorphism f_T homotopic to f with $K(f_T) = \inf\{K(\phi) \,|\, [\Sigma, f, \Sigma_1] = [\Sigma, \phi, \Sigma_1]\}$. That is,$\phi : \Sigma \to \Sigma_1$ is a quasiconformal homeomorphism which is homotopic to f. In a sense f_T comes closest to being conformal.

The solution is given by the non-trivial Teichmüller's theorem of which we now partially state statement.

Theorem A.3.10. *Given any $[\Sigma, f, \Sigma_1] \in T(\Sigma)$ there is a unique extremal f_T in the homotopy class of f solving Teichmüller's extremal problem.*

This can be formulated in terms of $T(G)$. Given $\mu \in L^\infty(U, G)_1$, there exist a unique $\mu_T \in L^\infty(U, G)_1$ such that $\|\mu_T\|_\infty \leq \|\mu_1\|_\infty$ for any μ_1 such that $[\mu_1] = [\mu] \in T(G)$.

Appendix B

Plemelj-Sokhotski Formula on Riemann Surfaces

In this appendix a generalization of the Plemelj-Sokhotski formula to Riemann surfaces will be stated along with a brief outline of the proof. The details can be found in Rodin [47] and Zverovich [52]. We first state the classical Plemelj-Sokhotski formula in the complex plane. Let γ be a smooth simple closed curve in $\mathbb{C}$, and let g be a sufficiently smooth (Hölder continuous) function from γ to $\mathbb{C}$. Let

$$F(z) = \frac{1}{2\pi i} \int \frac{g(w)}{w - z} dw$$

and note that F is analytic on $\mathbb{C} \setminus \gamma$. We will refer to $dw/(w-z)$ as the *Cauchy kernel*. The classical Plemelj-Sokhotski formula states

$$F^{+}(z) - F^{-}(z) = g(z)$$

for all $z \in \gamma$, where $F^{\pm}$ denotes the limits as the contour is approached from the inside or outside. See Gakov [19] for a proof.

Let L be a smooth curve (or curves) that separate a Riemann surface into two components. The theorem we are aiming to prove is that $F^{+} - F^{-} = g$ if and only if $\int_L g dZ_{\mu} = 0$, where dZ_{μ} is a basis for holomorphic differentials. The proof generalizes the method for genus-zero in that F will be represented by a Cauchy-type integral. The construction of the appropriate kernels relies on existence of abelian differentials of the first, second and third kinds with certain properties.

B.1 Background and Notation

A good reference for this material is Farkas and Kra [12]. We follow their book very closely in this section.

Notation

- M - Riemann Surface with genus g.
- a_j, b_j - Homology basis; $j = 1, \ldots g$ (a and b are dual curves)
- $\aleph_i$, $i = 1, \ldots, 2g$ is the same homology basis, first running over the a's and then the b's.
- Closed curves a and b that intersect exactly once are called *dual*
- $\mathcal{M}$ - Riemann surface represented by the polygon with $4g$ sides

$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \ldots$$

  with identification between a_1 and a_1^{-1} and so on.

Harmonic Differentials

For each (homologically non-trivial) cycle c, there exists an explicit construction for a harmonic differential η_c with the following properties: closed, not exact, smooth, real valued, supported on a neighborhood of c.

If c^* is a dual curve to c, then from the explicit construction it easily follows that

$$\int_{c^*} \eta_c = 1 \ .$$

In particular the integral is not zero which implies that η_c is not exact.

Meromorphic Differentials

To get exact differentials we must allow singularities. These will be able to be prescribed. Using an explicitly constructed function, Weyl's lemma (for harmonic functions) and a couples of pages of analysis gives the following result.

Theorem B.1.1 (Farkas & Kra, page 48). *Let z be a local coordinate vanishing at $P_0 \in M$. There exists a function u on M with the following properties.*

- *u is harmonic on $M \backslash \{P_0\}$*

- *$u - z^{-n}$ is harmonic on every sufficiently small neighborhood of P_0*

Let $\alpha = du$ and $\omega = \frac{-1}{2n}(\alpha + i \star \alpha)$. From the above theorem we see that ω is a meromorphic differential on M which is holomorphic on $M \backslash \{P\}$ and has a singularity $1/z^{n+1}$ at P.

Theorem B.1.2 (Farkas & Kra, page 49). *Let z_1 and z_2 be local coordinates vanishing at $P_1, P_2 \in M$. There a real-valued function u on M satisfying the following:*

- *u is harmonic on $M \backslash \{P_1, P_2\}$.*
- *$u - \log|z_1|$ is harmonic in a neighborhood of P_1.*
- *$u + \log|z_2|$ is harmonic in a neighborhood of P_2.*

Let $\alpha = du$ and $\omega = \alpha + i \star \alpha$. From the properties of u we see that ω is a meromorphic differential which is holomorphic on $M \backslash \{P_1, P_2\}$ with the only singularities being $1/z_1$ at P_1 and $-1/z_2$ at P_2.

Holomorphic Differentials

By using the construction for harmonic differentials above it is not too hard to show that the dimension of the vector space, H, of harmonic differentials has dimension $2g$. The idea is that for each element a in the homology basis we have a differential η_a.

Let $\alpha_1, \ldots, \alpha_{2g}$ be a basis for H. Recall that if α harmonic differential then $\omega = \alpha + i \star \alpha$ is harmonic. In fact H decomposes into the direct sum of the space of holomorphic differentials and the space of anti-holomorphic differentials. With a bit more work (really just linear algebra) it turns out that $\zeta_j = \alpha_j + i \star \alpha_j$ for $j = 1, \ldots, g$, is a basis for the vector space of holomorphic differentials. Hence the complex dimension is g. The basis is unique if we require

$$\int_{a_j} \zeta_k = \delta_{jk} \;, j, k = 1, \ldots, g$$

Note that we are only specifying half the periods.

Bilinear Relations

If θ and $\tilde{\theta}$ are closed differentials on M then

$$\iint_M \theta \wedge \tilde{\theta} = \sum_{j=1}^{g} \left[\int_{a_j} \theta \int_{b_j} \tilde{\theta} - \int_{b_j} \theta \int_{a_j} \tilde{\theta} \right] .$$

There are two ways to prove this, one of which is the following. Consider M as represented by the polygon $\mathcal{M}$ with identification. As $\mathcal{M}$ is simply connected, $\theta = df$ for some smooth function f. An application of Stokes theorem gives

$$\iint_M \theta \wedge \tilde{\theta} = \int_{\partial \mathcal{M}} f \tilde{\theta}$$

Then it is not too hard to show

$$\int_{\partial \mathcal{M}} f \tilde{\theta} = \sum_{j=1}^{g} \left[\int_{a_j} \theta \int_{b_j} \tilde{\theta} - \int_{b_j} \theta \int_{a_j} \tilde{\theta} \right] .$$

Note that this formula remains valid if $\tilde{\theta}$ is meromorphic.

Now assume that θ and $\tilde{\theta}$ are holomorphic. Then $\theta \wedge \tilde{\theta} = 0$ and so in particular the right hand side of the above equation is zero. If we set $\theta = \zeta_j$ and $\tilde{\theta} = \zeta_k$ the we obtain the following important formula

$$\int_{b_j} \zeta_k = \int_{b_k} \zeta_j .$$

The following is very important for our purposes.

Theorem B.1.3 (Farkas & Kra, page 66). *We can prescribe uniquely the a periods of a holomorphic differential. That is, by specifying the values of all the a periods*

$$\int_{a_j} \theta$$

we uniquely determine the holomorphic differential θ.

Abelian Differentials of the Third kind

Let $P, Q \in M$ and let τ_{PQ} be an abelian differential of the third kind with the following properties. Poles of first order at $P$and Q, residues of $+1$ at P and -1 at Q. The existence of such a differential was determined above. It is no longer true that the integral of τ around a closed curve depends only on the homology class of the curve. However, if c and c' are homologous curves, then it can be proved that

$$\int_c \tau - \int_{c'} \tau = 2\pi i n$$

This fact is very useful in generalizing the Plemelj-Sokhotski formula to arbitrary genus.

Let $\theta = \tau_{PQ}$ and $\tilde{\theta} = \tau_{RS}$. We cut $\mathcal{M}$by a curve from P to a point on $\partial\mathcal{M}$, and from this point to Q.Denote the cut surface by $\mathcal{M}'$. This gives a simply connected surface on which $\theta = df$ for some smooth f. Using the formulas above we get

$$\int_{\partial\mathcal{M}'} f\tilde{\theta} = 2\pi i \int_S^R \theta$$

By making a similar cut for the points R and S we get

$$\int_S^R \tau_{PQ} = \int_Q^P \tau_{RS}.$$

In the notation of Rodin [47] notation, which we explain below, this equality is

$$\omega_{p_0p}(q) - \omega_{p_0p}(q_0) = \omega_{q_0q}(p) - \omega_{q_0q}(p_0).$$

B.2 Plemelj-Sokhotski formula

In this section we follow Rodin [47], but to do that we first must change notation.

Dictionary:

- $z(p) = p$ - local coordinate at $p \in M$
- $\partial_p f = \frac{\partial f}{\partial z(p)} dz(p)$
- $d\omega_{q_1q_2}(p)$ corresponds to $\tau q_1 q_2$
- $\omega_{q_1q_2}(p)$ is a primitive of $d\omega_{q_1q_2}(p)$. If we specify $\omega_{q_1q_2}(p_0) = 0$, then the primitive is $\int_{p_0}^p d\omega_{q_1q_2}$.
- $dt^n_{p,z}(q)$ is a abelian differential of the second kind with principal part

$$\frac{-n dz(q)}{[z(q) - z(p)]^{n+1}}$$

Let

$$M^*(p,q)dp = \partial_p[\omega_{p_0p}(q) - \omega_{p_0p}(q_0)]$$

and we note (see Zverovich [52]) the key property that in the limit $p \to q$,

$$M(p,q)dp = \frac{dp}{p-q} + \text{regular terms.}$$

In this sense, $M(p,q)$ is analogous to the Cauchy kernel $dt/(z-t)$ in the complex plane.

Important properties of the kernel are:

- $M^*(p,q)dp$ is analytic(away from the poles) in p and q.
- The residues at $p = q, q_0$ are $+1$ and -1 respectively.
- As a function of q the kernel is an abelian integral of the second kind with a pole of first order at $q = p$.
- We can choose a single-valued branch on $\mathcal{M}$ by the condition $M^*(p, q_0) = 0$

Let L be a smooth curve and $g(p)$ a function on L that is at least Holder-continuous. (Note that the same conditions as are needed in planar case.) Let

$$f(q) = \frac{1}{2\pi i}\int_L g(p)M^*(p,q)dp.$$

The Plemelj-Sokhotski formula gives $f^+(q) - f^-(q) = g(q)$, where $f^\pm$ are the limiting values from the let and right hand sides of the curve L. The problem is that $f(q)$ is not single-valued on M. To make f single valued we need all the periods to be zero. The a periods are no problem as we can always remove these by the addition of a suitable holomorphic differential. To remove the b periods we must introduce additional singularities by adding abelian differentials of the second kind. In general the Riemann-Roch theorem is used to guarantee that we find such an abelian differential. The new kernel will be of the form

$$M(p,q)dp = M^*(p,q)dp - \sum_{k=1}^{g} t^1_{q_k} dZ_k(p)$$

where dZ_k is a basis for the space of holomorphic differentials. Let

$$F(q) = \frac{1}{2\pi i}\int_L g(p)M(p,q)dp\ .$$

We have not changed the nature of the singularity at $p = q$. However, $F(q)$ is not holomorphic on $M \backslash L$ because of the poles we have introduced at q_k. Additional calculations show that the singular part of $F(q)$ at these poles is given by

$$-\frac{1}{2\pi i}\int_L g(p)dZ_k(p) \ .$$

If these integrals are all zero then F will be analytic as required. We record this result in the following theorem. A proof can be found in Rodin [47].

Theorem B.2.1. *Let $\mathcal{M}$ be a compact Riemann surface and L be a finite set of smooth closed curve that separate $\mathcal{M}$ into two components. Let $g : L \to \mathbb{C}$ be a Hölder-continuous function.*

There exists a holomorphic function F on $\mathcal{M} \setminus L$ such that

$$F^+ - F^- = g$$

if and only if

$$\int_L g(p)dZ_k(p) = 0$$

where $k = 1, \ldots, g$ and dZ_k is a basis for the space of holomorphic differentials and $F^\pm$ are the limiting values of F as L is approached from the left and right.

Remark B.2.2. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be the components of $\mathcal{M}$ created by the separating curve(s) L. The function F is really two function F_1 and F_2 on $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively. Alternatively we think of F as a single function with a discontinuity along L. The value of the *jump* across L is given by the function g.

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Vita

David Aubrey Alfred Radnell

24/04/1974	Born in Cairns, Queensland, Australia.
1992 - 1995	University of Queensland, St. Lucia, QLD, Australia.
12/1994	B.Sc. (major in mathematics), University of Queensland.
12/1995	B.Sc.[Honours] (mathematics), University of Queensland.
1996 - 1997	The University of Adelaide, SA, Australia. PhD program in Mathematics.
1997 - 2003	Rutgers, The State University of New Jersey, NJ, USA. PhD Program in mathematics.
10/2003	Doctor of Philosophy in Mathematics.

1996 - 1997	Australian Postgraduate Award, The University of Adelaide.
1997 - 2003	Teaching/Graduate Assistant, Rutgers, The State University of New Jersey.
09/2003	T. H. Hildebrandt Assistant Professor, Department of Mathemaics, University of Michigan, MI, USA.