QUASICONFORMAL MAPS OF BORDERED RIEMANN SURFACES WITH L^2 BELTRAMI DIFFERENTIALS

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Abstract. Let Σ be a Riemann surface of genus g bordered by n curves homeomorphic to the circle $\mathbb{S}^1$. Consider quasiconformal maps $f : \Sigma \to \Sigma_1$ such that the restriction to each boundary curve is a Weil-Petersson class quasisymmetry. We show that any such f is homotopic to a quasiconformal map whose Beltrami differential is L^2 with respect to the hyperbolic metric on Σ. The homotopy $H(t, \cdot) : \Sigma \to \Sigma_1$ is independent of t on the boundary curves; that is $H(t, p) = f(p)$ for all $p \in \partial\Sigma$.

1. Introduction

The so-called Weil-Petersson class quasisymmetries of the unit circle $\mathbb{S}^1$ are those quasisymmetries whose corresponding conformal welding maps have pre-Schwarzians in the Bergman space. This class was studied for example by G. Cui [2], H. Guo [3], L. Takhtajan and L.-P. Teo [13] and Y. Hu and Y. Shen [4]. The Weil-Petersson class quasisymmetries of $\mathbb{S}^1$ (henceforth WP-class quasisymmetries) can also be characterized as those quasisymmetries which are the boundary values of quasiconformal maps with L^2 Beltrami differentials with respect to the hyperbolic metric [2, 3]. Y. Shen [10] showed that a homeomorphism h of the circle is a Weil-Petersson class quasisymmetry if and only if h is absolutely continuous and $\log h'$ is in the fractional $1/2$ Sobolev space. There has been growing interest recently in the so-called WP class universal Teichmüller space and its associated mappings (see the introduction to [10]).

Let Σ be a Riemann surface of genus g with n boundary curves, each of which is homeomorphic to $\mathbb{S}^1$. We assume that the boundary curves of Σ are borders (in the sense of Ahlfors and Sario [1]).

The main result of this paper is that every quasiconformal map of Σ with WP-class boundary values is homotopic to a quasiconformal map whose Beltrami differential is in L^2. We also show that if Σ and Σ_1 are bordered Riemann surfaces of genus g bordered by n homeomorphic circles, then given any collection of WP-class quasisymmetries $\phi_i : \partial_i\Sigma \to \partial_i\Sigma_1$ (where $\partial_i\Sigma, \partial_i\Sigma_1$ denote the enumerated boundary components of the surfaces respectively), there is a quasiconformal map $f : \Sigma \to \Sigma_1$ simultaneously extending the maps ϕ_i whose Beltrami differential is L^2 with respect to the hyperbolic metric. This generalizes a result of Cui for the disk [2] (circulated earlier in a pre-print). Guo [3] extended Cui's results to the L^p case for $p \geq 1$. We are grateful to the referee for clarifying the attribution of this result.

In order to prove these results, we use sewing techniques developed by two of the authors [7] which in turn require the lambda-lemma in the form given by Z. Słodkowski [11]. We also need a characterization of hyperbolically L^2 Beltrami differentials on bordered Riemann surfaces in terms of charts from doubly-connected neighbourhoods of the boundary curves

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into annuli. Namely, the norm in the chart with respect to the hyperbolic metric on a disk controls the L^2 estimate on the surface. In fact, this argument generalizes immediately to differentials of any order and any L^p spaces; thus we state and prove the general result.

The authors showed (see [8, 9]) that there is a natural Teichmüller space of bordered surfaces (which we called the refined or the WP-class Teichmüller space of bordered surfaces) which is a Hilbert manifold. An application of the results of the present paper shows that this Teichmüller space can be modelled by quasiconformal maps with hyperbolically L^2 Beltrami differentials.

2. WP-class maps

2.1. **WP-class quasisymmetries on $\mathbb{S}^1$.** In [8] the authors defined a Teichmüller space of bordered surfaces which possesses a Hilbert manifold structure. We briefly review some of the definitions and results, as well as introduce new definitions necessary in the next few sections.

Let

$$\mathbb{D} = \{z : |z| < 1\}, \quad \mathbb{D}^* = \{z : |z| > 1\} \cup \{\infty\}, \quad \text{and} \quad \bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}.$$

Let $A_1^2(\mathbb{D})$ denote the set of holomorphic differentials $h(z)dz$ on $\mathbb{D}$ such that

$$\iint_{\mathbb{D}} |h(z)|^2 dA < \infty$$

where dA denotes Lebesgue measure. That is, h is in the Bergman space of the disk. We use the notation $A_1^2(\mathbb{D})$ to be compatible with the notation for more general spaces of differentials which we will introduce in Section 3.1.

Definition 2.1. Let $\mathcal{O}_{\mathrm{WP}}^{\mathrm{qc}}$ denote the set of holomorphic one-to-one maps $f : \mathbb{D} \to \mathbb{C}$, with quasiconformal extensions to $\mathbb{C}$, such that $(f''(z)/f'(z))dz \in A_1^2(\mathbb{D})$ and $f(0) = 0$.

By results of Takhtajan and Teo, the image of $\mathcal{O}_{\mathrm{WP}}^{\mathrm{qc}}$ under the map

$$f \longmapsto \left(\frac{f''(z)}{f'(z)} dz, f'(0) \right) \tag{2.1}$$

is an open subset of the Hilbert space $A_1^2(\mathbb{D}) \oplus \mathbb{C}$ with the direct sum inner product [8, Theorem 2.3].

Elements of $\mathcal{O}_{\mathrm{WP}}^{\mathrm{qc}}$ arise as conformal maps associated to quasisymmetries in the following way. Given a quasisymmetry $\phi : \mathbb{S}^1 \to \mathbb{S}^1$, by the Ahlfors-Beurling extension theorem, there exists a quasiconformal map $w : \mathbb{D}^* \to \mathbb{D}^*$ such that $w|_{\mathbb{S}^1} = \phi$. This quasiconformal map has complex dilatation

$$\mu = \frac{\overline{\partial} f}{\partial f} \in L^\infty_{-1,1}(\mathbb{D}^*)$$

where $L^\infty_{-1,1}(\mathbb{D}^*)$ denotes the class of $(-1, 1)$ differentials with bounded essential supremum. Let f^μ be the solution to the Beltrami equation

$$\frac{\overline{\partial} f}{\partial f} = \hat{\mu}$$

where $\hat{\mu}$ is the Beltrami differential which equals μ on $\mathbb{D}^*$ and 0 on $\mathbb{D}$. We normalize f^μ so that $f^\mu(0) = 0$, $f^\mu(\infty) = \infty$ and ${f^\mu}'(\infty) = 1$ for definiteness. Let

$$f_\phi = f^\mu|_{\mathbb{D}}.$$

It is a standard result in Teichmüller theory that f_ϕ is independent of the choice of quasiconformal extension w, and furthermore $f_\phi = f_\psi$ if and only if $\phi = \psi$ [5, 6].

Definition 2.2. Let $\mathrm{QS}_{\mathrm{WP}}(\mathbb{S}^1, \mathbb{S}^1)$ denote the set of quasisymmetric mappings ϕ from $\mathbb{S}^1$ to $\mathbb{S}^1$ such that $f_\phi \in \mathcal{O}^{\mathrm{qc}}_{\mathrm{WP}}$.

We have the following theorem of Cui [2] (see also Guo [3]).

Theorem 2.3. *$\phi \in \mathrm{QS}_{WP}(\mathbb{S}^1, \mathbb{S}^1)$ if and only if ϕ has a quasiconformal extension $w : \mathbb{D}^* \to \mathbb{D}^*$ with Beltrami differential $\mu \in L^2_{-1,1}(\mathbb{D}^*)$.*

Here $L^2_{-1,1}(\mathbb{D}^*)$ denotes the set of $(-1, 1)$ differentials $\mu d\bar{z}/dz$ which are L^2 with respect to the hyperbolic metric on $\mathbb{D}^*$; that is, satisfying

$$\iint_{\mathbb{D}^*} \frac{|\mu(z)|^2}{(|z|^2 - 1)^2} dA < \infty$$

where again dA is Lebesgue measure (this is a special case of Definition 3.1 ahead). Note that since w is quasiconformal, its Beltrami differential automatically satisfies $\mu \in L^\infty_{-1,1}(\mathbb{D}^*)$.

Theorem 2.3 was generalized to the L^p case by Guo [3]. We also have the following recent remarkable result of Shen [10], which answers a question posed by Takhtajan and Teo [13, Remark 1.10].

Theorem 2.4. *A homeomorphism $\phi : \mathbb{S}^1 \to \mathbb{S}^1$ is in $\mathrm{QS}_{WP}(\mathbb{S}^1, \mathbb{S}^1)$ if and only if ϕ is absolutely continuous and $\log \phi' \in H^{1/2}(\mathbb{S}^1)$ where $H^{1/2}(\mathbb{S}^1)$ is the fractional $1/2$ Sobolev space.*

Although this result is not needed in this paper, we mention it because it provides a direct intrinsic characterization for $\mathrm{QS}_{\mathrm{WP}}(\mathbb{S}^1, \mathbb{S}^1)$.

2.2. **Quasiconformal maps of Riemann surfaces with WP-class boundary values.** From now on, let Σ be a Riemann surface of genus g bordered by n curves homeomorphic to $\mathbb{S}^1$; we will always assume that $n > 0$ when the term "bordered" is used. We clarify the meaning of "bordered" now. It is assumed that the Riemann surface is bordered in the sense of Ahlfors and Sario [1, II.1.3]. That is, the closure $\overline{\Sigma}$ of Σ is a Hausdorff topological space, together with a maximal atlas of charts from open subsets of $\overline{\Sigma}$ into relatively open subsets of the closed upper half plane in $\mathbb{C}$, such that the overlap maps are conformal on their interiors. (In particular, these charts have a continuous extension to the boundary). Thus for each point p on the boundary, there exists a chart from an open set U onto a disc $D = \{z : |z| < 1 \text{ and } \mathrm{Im}(z) > 0\}$ and a conformal map ϕ of U onto D, such that ϕ extends homeomorphically to a relatively open set $\hat{U} \subset \overline{\Sigma}$ which takes a segment of the boundary containing p in its interior to a line segment in the plane. We will refer to such a chart (ϕ, U) as an "upper half plane boundary chart". In order to avoid needless proliferation of notation, we will not distinguish ϕ notationally from its continuous extension, nor U from $\hat{U}$.

We will further allow charts in the interior of Σ which map onto open neighbourhoods of 0 in $\mathbb{C}$. We also allow boundary charts onto sets of the form $\{z : |z| \leq 1\} \cap \{z : |z - a| < r\}$ where $r < 1$ and $|a| = 1$ with conformal overlap maps as with the half-plane charts. We will refer to such a boundary chart as a "disc boundary chart". We refer to either a disc boundary chart or an upper half plane boundary chart as a "boundary chart".

Finally, when we say that Σ is bordered by n curves homeomorphic to $\mathbb{S}^1$, we mean that the boundary $\partial\Sigma$ consists of n connected components, each of which is homeomorphic to $\mathbb{S}^1$ when endowed with the relative topology inherited from $\overline{\Sigma}$. To say that Σ is of genus g

means that Σ is biholomorphic to a subset Σ^B of a compact Riemann surface $\widetilde{\Sigma}$ of genus g in such a way that the complement of $\overline{\Sigma^B}$ in $\widetilde{\Sigma}$ consists of n disjoint open sets biholomorphic to $\mathbb{D}$. Equivalently, the double of Σ has genus $2g+n$.

With this terminology established we may now make the following definition.

Definition 2.5. We say Σ is a bordered surface of type (g,n) if it is a bordered surface of genus g bordered by n boundary curves homeomorphic to $\mathbb{S}^1$, in the sense of the last three paragraphs.

We will also need one more kind of chart at the boundary. Let

$$\mathbb{A}_r = \{z \,:\, 1 < |z| < r\}.$$

The following proposition is elementary.

Proposition 2.6. *Let Σ be a bordered Riemann surface of genus g bordered by n curves $\partial_i\Sigma$, $i = 1,\ldots,n$, homeomorphic to $\mathbb{S}^1$. For each i, there exists an open set $A \subset \Sigma$ and an annulus $\mathbb{A}_r$ such that*

1. *$\partial_i\Sigma$ is contained in the closure of A,*
2. *$\partial A \cap (\partial_i\Sigma)^c$ is compactly contained in Σ, and*
3. *there is a conformal map $\zeta : A \to \mathbb{A}_r$ for some r.*

For any such A, $\mathbb{A}_r$, and ζ, ζ has a homeomorphic extension to $A \cup \partial_i\Sigma$.

Furthermore, A, r and ζ can be chosen so that $\partial A \backslash \partial_i\Sigma$ is an analytic curve. In that case ζ has a homeomorphic extension to the closure of A, which takes $\overline{A}$ onto the closed annulus $\overline{\mathbb{A}_r}$.

We call such a chart a "collar chart" of $\partial_i\Sigma$, and A a "collar" of $\partial_i\Sigma$.

We may now define WP-class quasisymmetries between boundary curves of bordered Riemann surfaces, as in ([8] or [9]).

Definition 2.7. Let Σ_1 and Σ_2 be bordered Riemann surfaces of type (g_i, n_i) respectively, and let C_1 and C_2 be boundary curves of Σ_1 and Σ_2 respectively. Let $\mathrm{QS}_{\mathrm{WP}}(C_1, C_2)$ denote the set of orientation-preserving homeomorphisms $\phi : C_1 \to C_2$ such that there are collared charts H_i of C_i, $i = 1, 2$ respectively, and such that $H_2 \circ \phi \circ H_1^{-1}\big|_{\mathbb{S}^1} \in \mathrm{QS}_{\mathrm{WP}}(\mathbb{S}^1, \mathbb{S}^1)$.

Remark 2.8. The notation $\mathrm{QS}_{\mathrm{WP}}(\mathbb{S}^1, C_1)$ will always be understood to refer to $\mathbb{S}^1$ as the boundary of an annulus $\mathbb{A}_r$ for $r > 1$. We will also write $\mathrm{QS}_{\mathrm{WP}}(\mathbb{S}^1) = \mathrm{QS}_{\mathrm{WP}}(\mathbb{S}^1, \mathbb{S}^1)$.

Remark 2.9. In [7, Section 2.4], Definition 2.7 was given with $\mathrm{QS}_{\mathrm{WP}}(\mathbb{S}^1)$ replaced by standard quasisymmetries $\mathrm{QS}(\mathbb{S}^1)$.

The following property of $\mathrm{QS}_{\mathrm{WP}}(C_1, C_2)$ ([8, 9]) verifies the naturality of Definition 2.7.

Proposition 2.10. *If $\phi \in \mathrm{QS}_{WP}(C_1, C_2)$ then for any pair of collar charts H_i of C_i, $i = 1, 2$ respectively, $H_2 \circ \phi \circ H_1^{-1}\big|_{\mathbb{S}^1} \in \mathrm{QS}_{WP}(\mathbb{S}^1)$.*

We will be concerned only with quasiconformal maps whose boundary values are in $\mathrm{QS}_{\mathrm{WP}}$. Any such quasiconformal mapping has a homeomorphic extension taking the closure of Σ_1 to the closure of Σ_2. This extension must map each boundary curve $\partial_i\Sigma_1$ homeomorphically onto a boundary curve $\partial_j\Sigma_2$.

Definition 2.11. Let Σ_1 and Σ_2 be bordered Riemann surfaces of type (g,n), with boundary curves C_1^i and C_2^j, $i = 1,\ldots,n$ and $j = 1,\ldots,n$, respectively. The class of maps $\mathrm{QC}_0(\Sigma_1, \Sigma_2)$

consists of those quasiconformal maps from Σ_1 onto Σ_2 such that the continuous extension to each boundary curve C_1^i, $i = 1, \ldots, n$ is in $\mathrm{QS}_{\mathrm{WP}}(C_1^i, C_2^j)$ for some $j \in \{1, \ldots, n\}$.

Remark 2.12. The continuous extensions to the boundary will be made without further comment. We will not make any notational distinction between a quasiconformal map f and its continuous extension.

For a quasiconformal map f let $\mu(f)$ denote its Beltrami differential as above. Theorem 2.3 above motivates the following definition.

Definition 2.13. Let $\mathrm{QC}_r(\Sigma, \Sigma_1)$ be the set of $f \in \mathrm{QC}_0(\Sigma, \Sigma_1)$ such that for any i and any collar chart $\zeta_i : A_i \to \mathbb{A}_{r_i}$ on a collar A_i of $\partial_i \Sigma$,

$$\iint_{\mathbb{A}_{r_i}} \frac{\left|\mu(f \circ \zeta_i^{-1})\right|^2}{(1 - |z|^2)^2} \, dA < \infty. \tag{2.2}$$

This condition can be thought of as requiring that $\mu(f)$ be "hyperbolically L^2 near $\partial_i \Sigma$". The condition appears to depend on the choice of chart, and it is not immediately obvious if this relates to whether or not the Beltrami differential $\mu(f)$ is in $L^2_{-1,1}(\Sigma)$. We will show that the conditions are the same. We will prove this in the next section, with the help of a general local characterization of hyperbolic L^p spaces.

3. L^p DIFFERENTIALS WITH RESPECT TO THE HYPERBOLIC METRIC

3.1. **Definition of the L^p spaces of differentials.** First we establish some notation for the various function spaces involved. It is convenient here to not have to refer directly to the lift, as is the usual practice. The (obviously) equivalent definitions can be found for example in [6].

Let Σ be a Riemann surface with a hyperbolic metric. Let $\mathcal{U}$ be an open covering of the Riemann surface Σ by open sets U, each of which possesses a local parameter $\phi_U : U \to \mathbb{C}$ compatible with the complex structure. For $k, l \in \mathbb{Z}$ a (k, l)-differential h is a collection of functions $\{h_U : \phi_U(U) \to \mathbb{C} \, : \, U \in \mathcal{U}\}$ such that whenever $U \cap V$ is non-empty, denoting by $z = g(w) = \phi_V \circ \phi_U^{-1}(w)$ the change of parameter, the functions h_U and h_V satisfy the transformation rule

$$h_V(w) g'(w)^k \overline{g'(w)}^l = h_U(z); \tag{3.1}$$

that is, h has the expression $h_U(z) dz^k d\bar{z}^l$ in local coordinates. For example, a Beltrami differential, or $(-1, 1)$ differential, is a collection of functions satisfying the transformation rule

$$h_V(w) \frac{\overline{g'(w)}}{g'(w)} = h_U(z); \tag{3.2}$$

that is, h has the expression $h_U(z) d\bar{z}/dz$ in local coordinates. Similarly a quadratic differential is a $(2, 0)$ differential and a function is a $(0, 0)$ differential.

We will be concerned with those differentials which are L^p with respect to the hyperbolic metric for some p (in this paper, we always have either $p = 1$, $p = 2$, or $p = \infty$). Denote the expression for the hyperbolic metric g in local coordinates by $\rho_U(z)^2 \, |dz|^2$ for a strictly positive function ρ_U; thus the metric transforms according to the rule

$$\rho_V(w) |g'(w)| = \rho_U(z). \tag{3.3}$$

Thus, if W is an open set, which we momentarily assume to be entirely contained in some $U \in \mathcal{U}$, for a (k,l)-differential we can define an L^p integral with respect to the hyperbolic metric by

$$\|h\|^p_{p,\Sigma,W} = \int_{\phi_U(W)} |h_U(z)|^p \rho_U(z)^{2-mp},$$

where $m = k + l$ and the right-hand integral is taken with respect to Lebesgue measure in the plane. It is easily checked that if W is entirely contained in another open set $V \in \mathcal{U}$, then the integral obtained using ϕ_V, h_V and ρ_V as above is identical, by (3.1), (3.3) and a change of variables.

By the standard construction using a partition of unity subordinate to the open cover $\mathcal{U}$, one can define the norm $\|h\|_{p,\Sigma,W}$ on any open set $W \subseteq \Sigma$, including $W = \Sigma$. Similarly, one can define an L^∞ norm

$$\|h\|_{\infty,\Sigma,W} = \| |h_U(z)| \rho_U^{-m} \|_\infty$$

for open sets W in a single chart where the right hand side is the standard essential supremum with respect to Lebesgue measure. As above this extends to any open subset $W \subseteq \Sigma$.

Definition 3.1. Let $W \subset \Sigma$ be an open set. For $1 \leq p \leq \infty$, let

$$L^p_{k,l}(\Sigma, W) = \{(k,l) - \text{differentials } \; h \, : \, \|h\|_{p,W} < \infty\}.$$

Let

$$A^p_k(\Sigma, W) = \{h \in L^p_{k,0}(\Sigma, W) \, : \, h \; \text{ holomorphic}\}.$$

Denote $L^p_{k,l}(\Sigma,\Sigma)$ by $L^p_{k,l}(\Sigma)$ and $A^p_k(\Sigma,\Sigma)$ by $A^p_k(\Sigma)$.

It will always be understood that any L^p norm arising here is with respect to the hyperbolic metric. Indeed, one cannot define the norm in general without the use of some invariant metric, except in special cases (e.g. for $k = 2$, $l = 0$ and $p = 1$).

Remark 3.2. We will not distinguish the norms $\| \cdot \|_{p,W}$ notationally with respect to the order of the differential, since the type of differential uniquely determines the norm. If the subscript "W" is omitted, it is assumed that $W = \Sigma$.

3.2. **Boundary characterization of hyperbolically L^p differentials.** In this section, we will show that the condition that a differential be hyperbolically L^p can be expressed locally in terms of the hyperbolic metric of a disk, collar, or half-chart. Denote the expression for the hyperbolic metric on the upper half plane by

$$\lambda_{\mathbb{H}}(z)^2 |dz|^2$$

where $\lambda_{\mathbb{H}}(z) = 1/\mathrm{Im}(z)$. Similarly on the disc the hyperbolic metric is

$$\lambda_{\mathbb{D}}(z)^2 |dz|^2$$

where $\lambda_{\mathbb{D}}(z) = 1/(1 - |z|^2)$.

Theorem 3.3. *Let Σ be a bordered Riemann surface of type (g,n) and let α be a (k,l)-differential on Σ. Fix $p \in [1,\infty]$. The following are equivalent.*

(1) $\alpha \in L^p_{k,l}(\Sigma)$.

(2) *For each point $q \in \overline{\Sigma}$, there is a chart (ϕ, U) of a neighbourhood of q into the upper half plane $\mathbb{H}$, such that if α is $h_U(z)dz^k d\bar{z}^l$ in local coordinates and $m = k + l$, the estimate*

$$
\begin{aligned}
&\iint_{\phi(U)} \lambda_{\mathbb{H}}^{2-mp}(z)|h_U(z)|^p < \infty, \qquad p \in [1,\infty) \\
&\|\lambda_{\mathbb{H}}(z)^{-m} h_U(z)\|_{\infty,\phi(U)} < \infty, \qquad p = \infty
\end{aligned} \tag{3.4}
$$

holds for the particular choice of p.

(3) *For each point $q \in \overline{\Sigma}$, there is a chart (ϕ, U) of a neighbourhood of q into the unit disc $\mathbb{D}$, such that if α is $h_U(z)dz^k d\bar{z}^l$ in local coordinates and $m = k + l$, the estimate*

$$
\begin{aligned}
&\iint_{\phi(U)} \lambda_{\mathbb{D}}^{2-mp}(z)|h_U(z)|^p < \infty, \qquad p \in [1,\infty) \\
&\|\lambda_{\mathbb{D}}(z)^{-m} h_U(z)\|_{\infty,\phi(U)} < \infty, \qquad p = \infty
\end{aligned} \tag{3.5}
$$

holds for the particular choice of p.

(4) *For each boundary curve $\partial_i\Sigma$, there is a collar chart (ϕ, U) of $\partial_i\Sigma$ for which the estimate (3.5) holds.*

(5) *For any collar chart (ϕ_i, U_i) of any boundary curve $\partial_i\Sigma$ the estimate (3.5) holds.*

Remark 3.4. As the reader may have noticed, there is no need to weight with the factor $(1 - |z|^2)$ in the interior, analytically. This is because $(1 - |z|^2)$ and ρ_U are continuous and the weight has no effect on the integral or on the L^∞ norm. Thus Theorem 3.3 is in effect about boundary values. However, for purely stylistic reasons, we shall keep the formulation of the theorem as above.

This theorem follows from an elementary estimate which we state as two lemmas.

Lemma 3.5. *Let Σ be a bordered Riemann surface of type (g, n). Let $q \in \overline{\Sigma}$. There is a chart (ζ, U) in a neighbourhood of q, with the following property. There is a disc $D \subset \zeta(U)$ centred on $\zeta(p)$ if $q \in \Sigma$ or a relatively open half-disc in $\overline{\mathbb{H}}$ centred on $\zeta(q)$ if $q \in \partial\Sigma$, and a $K > 0$ such that*

$$\frac{1}{K} \leq \left| \frac{\rho_U(z)}{\lambda_{\mathbb{H}}(z)} \right| \leq K \tag{3.6}$$

for all $z \in D$. Here $\rho_U(z)|dz|^2$ is the expression for the hyperbolic metric on Σ in the local parameter. The same claim holds for the hyperbolic metric $\lambda_{\mathbb{D}}$ on $\mathbb{D}$ and disk charts.

Proof. If $q \in \Sigma$, then choosing U to be an open neighbourhood of q with compact closure in Σ, the estimate follows immediately from the fact that ρ_U and $\lambda_{\mathbb{H}}$ are continuous and non-vanishing.

Fix $q \in \partial\Sigma$. First we show that there is at least one chart in which the claim holds. Let $\pi : \mathbb{H} \to \Sigma$ be the covering of Σ by the upper half plane. There is a relatively open set $\hat{U}$ in $\overline{\Sigma}$ containing q such that there is a single-valued branch $\phi = \pi^{-1}$ on $U = \hat{U} \cap \Sigma$. $\phi = \pi^{-1}$ is an isometry so that we have $\rho_U(z) = \lambda_{\mathbb{H}}(z)$ and thus the claim holds for this chart.

Let (ζ, V) be any other chart in a neighbourhood of q; we may assume without loss of generality that (ζ, V) is an upper half plane boundary chart centred on q. Let $H = \phi \circ \zeta^{-1}$ on $\zeta(U \cap V)$. In that case H maps an open interval on $\mathbb{R}$ containing $\zeta(q)$ to an open interval of $\mathbb{R}$ containing $\phi(q)$, so by Schwarz reflection H has an analytic continuation to an open disc containing an open interval on $\mathbb{R}$ with $\zeta(q)$ in its interior. Similarly the same claim holds for H^{-1}. We have

$$\frac{\rho_V(z)}{\lambda_{\mathbb{H}}(z)} = \frac{\rho_U(H(z))|H'(z)|}{\lambda_{\mathbb{H}}(z)} = \frac{\rho_U(H(z))}{\lambda_H(H(z))}\frac{\lambda_{\mathbb{H}}(H(z))|H'(z)|}{\lambda_{\mathbb{H}}(z)}$$

so it suffices to estimate $\lambda_{\mathbb{H}} \circ H|H'|^2/\lambda_{\mathbb{H}}$.

Let $w = H(z) = u(z) + iv(z)$ for real functions u and v, and let $z = x + iy$. We have that the hyperbolic metric is $\lambda_{\mathbb{H}}(z) = 1/y$. Since H is a biholomorphism, $H' \neq 0$. We claim that $v_y \neq 0$ at $\zeta(q)$. If not, we would have $u_x = v_y = 0$ at $\zeta(q)$. Furthermore since H maps an interval on $\mathbb{R}$ containing $\zeta(q)$ to an interval in $\mathbb{R}$, $v = 0$ on this interval so $u_y = -v_x = 0$ on an interval containing $\zeta(q)$. Thus the Jacobian of H at $\zeta(q)$ is zero, a contradiction. We conclude that there is a neighbourhood of $\zeta(q)$ on which $v_y \neq 0$. Using a Taylor series approximation in two variables, and the fact that $v(x, 0) = v_x(x, 0) = 0$, we have that

$$C|y| \leq |v(x, y)| \leq D|y| \tag{3.7}$$

for some constants $C, D > 0$ on some open disc centred on $\zeta(p)$ whose closure is contained in the domain of H. Furthermore since H is a biholomorphism there are constants $0 < E, F$ such that $E \leq H' \leq F$ on a possibly smaller open disc whose closure is contained in the domain of H. Since $\lambda_{\mathbb{H}} \circ H(z) = 1/v(z)$, by (3.7) there is a $K > 0$ such that

$$\frac{1}{K} \leq \frac{\lambda_{\mathbb{H}} \circ H|H'|}{\lambda_{\mathbb{H}}} \leq K$$

on this disk. This proves the claim.

The estimate for $\lambda_{\mathbb{D}}$ can be easily obtained by applying a Möbius transformation. □

Lemma 3.5 can be improved slightly to the following.

Lemma 3.6. *Let Σ be a bordered Riemann surface of type (g, n) and let (ζ_i, U_i) be a collar chart of $\partial_i\Sigma$. There is an annulus $\mathbb{A}_{r,1} \subseteq \zeta_i(U_i)$ with $\mathbb{A}_{r,1} := \{z;\, r < |z| < 1\}$ such that*

$$\frac{1}{K} \leq \left|\frac{\rho_{U_i}(z)}{\lambda_{\mathbb{D}}(z)}\right| \leq K$$

for all $z \in \mathbb{A}_{r,1}$.

Proof. Repeating the proof of Lemma 3.6, for every point $q \in \partial_i\Sigma$ one obtains an open half-disc $\{z : |z - \zeta_i(q)| < s_q\} \cap \mathbb{D}$ on which the estimate holds. Since $\partial_i\Sigma$ is compact the claim follows. □

Proof. (of Theorem 3.3) To see that (2) implies (1), observe that by Lemma 3.5 and the fact that $\overline{\Sigma}$ is compact, there is a finite collection of charts (ζ_i, U_i) and discs or half-discs D_i in $\mathbb{H}$ such that $\zeta_i^{-1}(D_i)$ cover Σ and on which the estimate (3.6) holds. Thus there are constants $C_i(m, p)$ such that $\rho_{U_i}(z)^{2-mp} \leq C_i(m, p)\lambda_{\mathbb{H}}(z)^{2-mp}$ for $p \in [1, \infty)$ and $\rho_{U_i}(z)^{-m} \leq C_i(m, \infty)\lambda_{\mathbb{H}}(z)^{-m}$. Thus for $p \in [1, \infty)$

$$\iint_{D_i} \rho_{U_i}(z)^{2-mp}|h_{U_i}(z)|^p \leq \iint_{D_i} C_i(m, p)\lambda_{\mathbb{H}}(z)^{2-mp}|h_{U_i}(z)|^p < \infty$$

for all i and for $p = \infty$

$$\|\rho_{U_i}(z)^{-m}h_{U_i}(z)\|_{\infty,D_i} < C_i(m, \infty)\|\lambda_{\mathbb{H}}(z)^{-m}h_{U_i}(z)\|_{\infty,D_i} < \infty.$$

Choosing a partition of unity subordinate to the finite covering proves (1).

Now we show that (2) follows from (1). For any point q let (ζ, V) be any chart in a neighbourhood of q, and let D be as in Lemma 3.5. We then have that there are constants $C(m,p)$ such that on D, $\lambda_{\mathbb{H}}(z)^{2-mp} \leq C(m,p)\rho_U(z)^{2-mp}$ for $p \in [1,\infty)$ and $\lambda_{\mathbb{H}}(z)^{-m} \leq C(m,\infty)\rho_U(z)^{-m}$ for $p = \infty$. Set $U = \zeta^{-1}(D)$ now and let ϕ be the restriction of ζ to U; we then have

$$\iint_{\phi(U)} \lambda_{\mathbb{H}}^{2-mp}(z)|h_U(z)|^p < C(m,p) \iint_{\phi(U)} \rho_U^{2-mp}(z)|h_U(z)|^p \leq \|\alpha\|_{p,\Sigma} < \infty$$

in the case that $p \neq \infty$ and

$$\|\lambda_{\mathbb{H}}(z)^{-m} h_U(z)\|_{\infty,\phi(U)} < \|\rho_U(z)^{-m} h_U(z)\|_{\infty,\phi(U)} \leq \|\alpha\|_{\infty,\Sigma} < \infty$$

in the case that $p = \infty$.

The equivalence of (3) and (1) follows from an identical argument. Clearly (5) implies (4) and (4) implies (3); on the other hand, if (1) holds, an argument similar to the proof of (2) above using Lemma 3.6 establishes (5) (note that by definition the inner boundary of a collar chart is compactly contained in Σ). □

Finally, we will need the following lemma explicitly separating out the contribution of the collar to the L^2 norm. We will only need the $p = 2$ case, but since the general case requires no extra work, we will state it in general.

Lemma 3.7. *Let Σ be a bordered Riemann surface of genus g with n boundary curves. Fix $p \in [1,\infty)$. Let (ζ, U) be a collection of collar charts (ζ_i, U_i) into $\mathbb{D}$ for each boundary $i = 1, \ldots, n$. There exist annuli $\mathbb{A}_{r_i,1} = \{z : r_i < |z| < 1\} \subset \zeta_i(U_i)$ such that $|z| = r_i$ is compactly contained in $\zeta_i(U_i)$, a compact set M such that*

$$M \cup \zeta_1^{-1}(\mathbb{A}_{r_1,1}) \cup \cdots \cup \zeta_n^{-1}(\mathbb{A}_{r_n,1}) = \Sigma,$$

and constants a and b_i such that for any $\alpha \in L^p_{k,l}(\Sigma)$

$$\|\alpha\|_p \leq a\|\alpha\|_{\infty,M} + \sum_{i=1}^n b_i \left(\iint_{\mathbb{A}_{r_i,1}} \lambda_{\mathbb{D}}^{2-mp}(z)|\alpha_{U_i}(z)|^p \right)^{1/p}.$$

The constants b_i depend only on the collar charts (ζ, U), r_i, p, k and l (not on α), and a^p is the hyperbolic area of M.

Proof. Once the annuli are chosen, one need only choose M such that M together with $\zeta_i^{-1}(\mathbb{A}_{r_i,1})$ cover Σ. The estimates on $\zeta_i^{-1}(\mathbb{A}_{r_i,1})$ follow from Lemma 3.6, as in the proof of Theorem 3.3.

The estimate on M is obtained as follows. Let (ξ_j, W_j), $j = 1, \ldots, N$ be charts into $\mathbb{D}$, such that the open sets W_j form an open cover of M. Let χ_j be a partition of unity of M subordinate to this covering (that is, $\sum \chi_j = 1$ on M and χ_j are supported in W_j). Let $\mathbb{1}_M$ denote the characteristic function of M. Then

$$\begin{aligned}
\|\alpha\|_{p,M}^p &= \sum_{j=1}^N \iint_{W_j} \chi_j \mathbb{1}_M |\alpha_{W_j}(z)|^p \rho_{W_j}(z)^{2-mp} \\
&\leq \sum_{j=1}^N \|\alpha\|_{\infty,W_j}^p \iint_{W_j} \chi_j \mathbb{1}_M \cdot \rho_{W_j}(z)^2 \\
&\leq a^p \|\alpha\|_{\infty,M}^p.
\end{aligned}$$

Since $p \geq 1$ the claim follows from the elementary inequality $\left(\sum a_k\right)^{1/p} \leq \sum a_k^{1/p}$. □

4. Homotopy classes of quasiconformal maps with WP-class boundary values

With the aid of the local characterization of hyperbolic L^p spaces in Section 3.2, we can now generalize Theorem 2.3 of Cui. First, we make the following definition.

Definition 4.1. Let

$$\mathrm{BD}_2(\Sigma) = \{\mu \in L^{\infty}_{-1,1}(\Sigma) \cap L^2_{-1,1}(\Sigma) \, : \, \|\mu\|_{\infty,\Sigma} \leq K \ \text{ for some } \ K < 1\}.$$

We thus have that

Theorem 4.2. *Let Σ and Σ_1 be bordered Riemann surfaces of type (g,n). Let $f : \Sigma \to \Sigma_1$ be quasiconformal with Beltrami differential $\mu(f)$. Then $f \in QC_r(\Sigma, \Sigma_1)$ if and only if $\mu(f) \in \mathrm{BD}_2(\Sigma)$.*

Proof. This follows directly from Theorem 3.3. □

We now prove some results relating to the existence of elements of QC_r in a fixed homotopy class. They ultimately rely on the extended lambda-lemma [11], through [7, Lemma 4.2].

Theorem 4.3. *Let Σ, Σ_1 be bordered Riemann surfaces of type (g,n) and $f : \Sigma \to \Sigma_1$ be a quasiconformal map. Let $\phi_i \in \mathrm{QS}_{WP}(\partial_i\Sigma, \partial_i\Sigma_1)$ for $i = 1, \ldots, n$ where $\partial_i\Sigma_1 = f(\partial_i\Sigma)$. There is a quasiconformal map $\hat{f} : \Sigma \to \Sigma_1$ in $\mathrm{QC}_0(\Sigma, \Sigma_1)$ such that $\hat{f}$ is homotopic to f and*

$$\hat{f}\Big|_{\partial_i\Sigma} = \phi_i.$$

The homotopy $G(t,z)$ can be chosen so that for each t, $G(t,\cdot)$ is a quasiconformal map.

Proof. For each i choose collar charts (ζ_i, U_i) of $\partial_i\Sigma$ onto $\mathbb{A}_{r_i}$, say, and (η_i, V_i) of $\partial_i\Sigma$ onto $\mathbb{A}_{s_i}$. For each i, choose numbers $R_i \in (1, r_i)$ and $S_i \in (1, s_i)$. Let $\gamma_i = \zeta_i^{-1}(|z| = R_i)$ and $\beta_i = f(\gamma_i)$. The map $\eta_i \circ f \circ \zeta_i^{-1}$ is a quasiconformal mapping from $\mathbb{A}_{R_i}$ onto a double connected domain A, whose inner boundary is $\{z : |z| = 1\}$ and whose outer boundary is the quasicircle $\eta_i(\beta_i)$. By [7, Theorem 2.13(2)], the restriction of $\eta_i \circ f \circ \zeta_i^{-1}$ to $|z| = R_i$ is a quasisymmetry (in the sense of Remark 2.9) onto $\eta_i(\beta_i)$.

By [7, Corollary 4.1] there is a quasiconformal mapping from A onto itself which is the identity on $\eta_i(\beta_i)$ and equals $\eta_i \circ \phi_i \circ f^{-1} \circ \eta_i^{-1}$ on $|z| = 1$. In fact, the proof of [7, Lemma 4.2 and Corollary 4.1] shows that this map can be embedded in a holomorphic motion $h_i : \Delta \times \overline{A} \to \overline{A}$ (in particular, a homotopy of quasiconformal maps) such that $h_i(1,z) = \eta_i \circ \phi_i \circ f^{-1} \circ \eta_i^{-1}$ for $|z| = 1$, $h_i(0,z) = z$ for all $z \in \eta_i(\beta_i)$, and $h_i(t,z) = z$ for $(t,z) \in [0,1] \times \eta_i(\beta_i)$. Setting $H_i(t,z) = h(t, \eta_i \circ f_i \circ \zeta_i^{-1}(z))$ we have a homotopy $H_i : [0,1] \times \overline{\mathbb{A}_{R_i}} \to \overline{A}$ such that

1. for each $t \in [0,1]$, $H_i(t,\cdot)$ is a quasiconformal homeomorphism,
2. $H_i(0,z) = \eta_i \circ f \circ \zeta_i^{-1}(z)$ for all z,
3. $H_i(1,z) = \eta_i \circ \phi_i \circ \zeta_i^{-1}(z)$ for $|z| = 1$, and
4. $H(t,z) = \eta_i \circ f \circ \zeta_i^{-1}(z)$ for $|z| = R_i$.

Next, we lift these homotopies back to the surfaces Σ and Σ_1 by composing with the charts, and sew them to f. Explicitly, we define the map $G : [0,1] \times \Sigma \to \Sigma_1$ by

$$G(t,q) = \begin{cases} \eta_i^{-1} \circ H(t, \zeta_i(q)), & q \in \zeta_i(A) \\ f, & \text{otherwise.} \end{cases}$$

One can verify that G is continuous on the seams γ_i by chasing compositions. By removability of quasicircles (see [12, Theorem 3]) $G(t,\cdot)$ is in fact a quasiconformal homeomorphism for each $t \in [0,1]$ and in particular a homotopy. Furthermore $G(0,z) = f(z)$ for all $z \in \Sigma$ and $G(1,z) = \phi_i(z)$ for all $z \in \partial_i\Sigma$ and $i = 1,\ldots,n$. This concludes the proof. □

We say that two quasiconformal maps $f : \Sigma \to \Sigma_1$ and $\hat{f} : \Sigma \to \Sigma_1$ are homotopic rel boundary if there is a homotopy from f to $\hat{f}$ which is constant on $\partial\Sigma$. (In particular, f and $\hat{f}$ are equal on $\partial\Sigma$.)

Theorem 4.4. *Let $f : \Sigma \to \Sigma_1$. $f \in \mathrm{QC}_0(\Sigma,\Sigma_1)$ if and only if f is homotopic rel boundary to some $\hat{f} \in \mathrm{QC}_r(\Sigma,\Sigma_1)$.*

Proof. Let (ζ_i, U_i) be a collar chart of each boundary curve $\partial_i\Sigma$ onto $\mathbb{A}_{r_i}$, and similarly (η_i, V_i) a collar chart of each boundary curve $\partial_i\Sigma_1$ onto $\mathbb{A}_{s_i}$. We may arrange that $U_i \cap U_j$ is empty for $i \neq j$ and similarly for the sets V_i. Let ϕ_i be the restriction of f to $\partial_i\Sigma$. By Theorem 2.3, there is a quasiconformal extension ψ_i of $\eta_i \circ \phi_i \circ \zeta_i^{-1}$ to $\mathbb{D}^*$ such that its Beltrami differential $\mu(\psi_i) \in \mathrm{BD}_2(\mathbb{D}^*)$. The idea of the rest of the proof is to "patch" $\eta_i^{-1} \circ \psi \circ \zeta_i$ together with f (note it agrees with f on $\partial_i\Sigma$); since $\mu(\psi_i) \in \mathrm{BD}_2(\mathbb{D}^*)$ the resulting map will be in $\mathrm{QC}_r(\Sigma,\Sigma_1)$ by definition.

We now proceed with the patching argument. Let T_i be a real number such that $1 < T_i < r_i$ and small enough that $f \circ \zeta_i^{-1}(|z| = T_i)$ is in V_i. Since ψ_i is a quasiconformal homeomorphism, there exists an R_i such that $1 < R_i < T_i$ and for all $|z| = R_i$

$$1 < \psi_i(z) < \min_{w \in \eta_i \circ f \circ \zeta_i^{-1}(|z|=T_i)} |w|.$$

In other words, the quasicircles $|w| = 1$, $\psi_i(|z| = R_i)$ and $\eta_i \circ f \circ \zeta_i^{-1}(|z| = T_i)$ are concentric. Let B_i denote the domain bounded by the latter two curves. Denote by A_i the annulus $R_i < |z| < T_i$. By [7, Corollary 4.1], there is a quasiconformal map $h : A_i \to B_i$ such that $h(z) = \psi_i(z)$ for $|z| = R_i$ and $h(z) = \eta_i \circ f \circ \zeta_i^{-1}(z)$ for $|z| = T_i$. Let

$$\tilde{h}(z) = \begin{cases} \psi_i(z), & z \in \mathbb{A}_{R_i} \\ h(z), & z \in A_i. \end{cases}$$

By removability of quasicircles [12, Theorem 3], this extends to a quasiconformal map of $\mathbb{A}_{T_i}$ onto $\eta_i \circ f \circ \zeta_i^{-1}(\mathbb{A}_{T_i})$.

Since $\tilde{h}$ agrees with $\eta_i \circ f_i \circ \zeta_i^{-1}$ on the boundary of the annulus $\mathbb{A}_{T_i}$, we have that they are homotopic rel boundary up to a $\mathbb{Z}$ action. Thus by composing with a quasiconformal map $g : A_i \to A_i$ such that g is the identity on $\partial\mathbb{A}_{T_i}$ we can arrange that $\tilde{h}$ is homotopic rel boundary to $\eta_i \circ f \circ \zeta_i^{-1}$. Let $H : [0,1] \times \mathbb{A}_{T_i} \to \eta_i \circ f \circ \zeta_i^{-1}(\mathbb{A}_{T_i})$ be such a homotopy. The important properties of H are that

$$\begin{aligned} H(0,z) &= \eta_i \circ f \circ \zeta_i^{-1}(z), && z \in \mathbb{A}_{T_i} \\ H(1,z) &= \psi_i(z), && z \in \mathbb{A}_{R_i} \\ H(t,z) &= \eta_i \circ f \circ \zeta_i^{-1}(z), && |z| = T_i,\ t \in [0,1]. \end{aligned}$$

Now we define $G : [0,1] \times \Sigma \to \Sigma_1$ by

$$G(t,z) = \begin{cases} \eta_i^{-1} \circ H_i(t, \zeta_i(z)), & z \in \zeta^{-1}(\mathbb{A}_{T_i}) \\ f(z), & \text{otherwise.} \end{cases}$$

This extends to a quasiconformal map from Σ to Σ_1 by removability of quasicircles. Chasing compositions we see that $G(0,z) = f(z)$ and $G(1,z) = \eta_i^{-1} \circ \psi_i \circ \zeta_i(z)$. By construction $\hat{f}(z) = G(1,z) \in \mathrm{QC}_r(\Sigma, \Sigma_1)$. □

Remark 4.5. As in the previous theorem, the proof shows that the homotopy can be chosen so that for each t, $G(t,z)$ is a quasiconformal map.

Remark 4.6. Since by Theorem 4.2 elements of $\mathrm{QC}_r(\Sigma)$ have Beltrami differentials in $\mathrm{BD}_2(\Sigma)$, this establishes the claim in the abstract and introduction: every quasiconformal map with WP-class boundary values is homotopic rel boundary to a quasiconformal map with square integrable Beltrami differential.

This also establishes

Corollary 4.7. *For bordered Riemann surfaces* Σ *and* Σ_1 *of type* (g,n)

$$\mathrm{QC}_r(\Sigma, \Sigma_1) \subseteq \mathrm{QC}_0(\Sigma, \Sigma_1).$$

Furthermore Theorem 4.4 immediately implies the following improvement of Theorem 4.3.

Corollary 4.8. *Theorem 4.3 holds with* $\mathrm{QC}_0(\Sigma, \Sigma_1)$ *replaced by* $\mathrm{QC}_r(\Sigma, \Sigma_1)$.

Since any two bordered Riemann surfaces of type (g,n) are quasiconformally equivalent, we also have the following result.

Theorem 4.9. *Let* Σ *and* Σ_1 *be bordered Riemann surfaces of type* (g,n)*. Let* $\phi_i : \partial_i\Sigma \to \partial_i\Sigma_1$ *be quasisymmetric maps for* $i = 1, \ldots, n$*. Then* $\phi_i \in \mathrm{QS}_{WP}(\partial_i\Sigma, \partial_i\Sigma_1)$ *for all* $i = 1, \ldots, n$ *if and only if there is a quasiconformal map* $f \in \mathrm{QC}_r(\Sigma, \Sigma_1)$ *such that*

$$f|_{\partial_i\Sigma} = \phi_i.$$

Proof. Assume that there is a quasiconformal map $f \in \mathrm{QC}_r(\Sigma, \Sigma_1)$ whose restriction to each ith boundary curve is ϕ_i. By Corollary 4.7 $f \in \mathrm{QC}_0(\Sigma, \Sigma_1)$. Thus by Definition 2.11 $\phi_i \in \mathrm{QS}_{\mathrm{WP}}(\partial_i\Sigma, \partial_i\Sigma_1)$.

Assume now that $\phi_i \in \mathrm{QS}_{\mathrm{WP}}(\partial_i\Sigma, \partial_i\Sigma_1)$ for $i = 1, \ldots, n$. Since Σ and Σ_1 are both of type (g,n), there is a quasiconformal map $f : \Sigma \to \Sigma_1$. By Corollary 4.8 there is a map $\hat{f} \in \mathrm{QC}_r(\Sigma, \Sigma_1)$ whose restriction to $\partial_i\Sigma$ is ϕ_i. □

This theorem can be considered to be a generalization of Theorem 2.3 above due to Cui.

Remark 4.10. One might be tempted to suppose that Theorem 4.9 follows from 2.3 by lifting to the universal cover. However it is not the case that the Beltrami differential of a quasiconformal map f in $\mathrm{QC}_r(\Sigma, \Sigma_1)$ has a lift in $L^2_{-1,1}(\mathbb{D}^*)$ (recall this notation refers to differentials which are L^2 with respect to $\lambda_{\mathbb{D}}(z)|dz|^2$). In fact, unless the Beltrami differential is zero almost everywhere, and hence f is conformal, the integral over any fundamental domain is non-zero. By the invariance of the lifted differential, the L^2 norm over $\mathbb{D}^*$ must be the sum over all fundamental domains of the L^2 norm on Σ, and is therefore infinite. In summary, unless f is conformal, the lifted Beltrami differential is not in $L^2_{-1,1}(\mathbb{D}^*)$. Thus there is no "lifted" version of Cui's theorem.

Finally, we give an application of these results. In [9] the authors defined a Teichmüller space of bordered Riemann surfaces as follows.

Definition 4.11. Let Σ be a bordered Riemann surface of genus g with n boundary curves. The Weil-Petersson class Teichmüller space of Σ is

$$T_{WP}(\Sigma) = \{(\Sigma, f, \Sigma_1) \,:\, f \in \mathrm{QC}_0(\Sigma, \Sigma_1)\}/\sim$$

where $(\Sigma, f_1, \Sigma_1) \sim (\Sigma, f_2, \Sigma_2)$ if and only if there is a biholomorphism $\sigma : \Sigma_1 \to \Sigma_2$ such that $f_2^{-1} \circ \sigma \circ f_1$ is homotopic to the identity rel boundary.

By Theorem 4.4, the following is immediately seen to be an equivalent formulation.

Theorem 4.12. *Let Σ be a bordered Riemann surface of genus g with n boundary curves. The Weil-Petersson class Teichmüller space of Σ can be expressed*

$$T_{WP}(\Sigma) = \{(\Sigma, f, \Sigma_1) \,:\, f \in \mathrm{QC}_r(\Sigma, \Sigma_1)\}/\sim$$

where $(\Sigma, f_1, \Sigma_1) \sim (\Sigma, f_2, \Sigma_2)$ if and only if there is a biholomorphism $\sigma : \Sigma_1 \to \Sigma_2$ such that $f_2^{-1} \circ \sigma \circ f_1$ is homotopic to the identity rel boundary.

Equivalently

$$T_{WP}(\Sigma) = \{(\Sigma, f, \Sigma_1) \,:\, \mu(f) \in \mathrm{BD}_2(\Sigma)\}/\sim .$$

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